



Geometrically nonlinear analysis of structures using stability functions and the tangent stiffness method

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Abstract

In this study, the application of Stability Functions and Tangent Stiffness Methods has been used to determine the geometrically nonlinear analysis of steel structures. The stability functions method is an iterative method that takes into account the axial forces occurring in the structural elements. The majority of the methods developed for the nonlinear analysis of structures are successive approximation methods. Stability functions have been used in the calculation of second-order effects accurately and in the consideration of geometrically nonlinear behavior with an iterative technique. In the tangent stiffness method, the nonlinear behavior of the structure has been obtained using iterative linear elastic analyses. In each iteration, the equilibrium equations have been rewritten in the deformed position of the structure, the geometric stiffness matrix has been established, and the compensation load vector has been applied to the system to balance the joint displacements. Both developed algorithms are coded with the Visual Basic .NET programming language. To demonstrate the application of the algorithms, the analysis of a 2-plane frame and 1-space truss system with unhindered lateral stability is performed, and the results are compared. In addition, the analysis and comparison of the same problems are made with the "P-Delta plus Large Displacements" option in SAP2000 software. The obtained results show good agreement with the benchmark solutions and the SAP2000 analyses, thereby confirming the accuracy and reliability of the proposed computational algorithms for geometrically nonlinear structural analysis.

1. Introduction

Engineering problems could be solved only by analytical methods for several centuries. In addition to analytical methods, numerical methods have recently found a large application area with the use of computers. As computational techniques turn to numerical analysis methods, nonlinear analysis solutions have become quite widespread. With the recent implementation of design specifications based on the ultimate strength of structures, it has been observed that the use of nonlinear analysis methods in structural design has increased.

Nonlinear complex problems that can be idealized with linear behavior are the easiest to analyze. Therefore, linearization, one of the most effective methods, is used in structural mechanics as well as in most engineering fields. As the behavior of the system moves away from linear behavior, the errors in the analysis resulting from linearization increase.

Two types of nonlinearities occur in structural problems. The first type is called material nonlinearity and results from the nonlinear elastic and plastic or viscoelastic behavior of the structural material. The second type is called geometric nonlinearity and occurs when the displacements are large enough to cause significant changes in the geometry of the structure. In this case, the equilibrium equations must be formulated for the deformed configuration of the structure. Geometric nonlinear analysis provides important information about the stability behavior of structures that linear analysis cannot access [1].

It is seen that the analysis and design of structural behavior are still made according to elastic theory. In these analyses made with elastic theory, which is a classical theory, second-order effects are neglected. Analyses made by ignoring the real behavior of structural elements are unrealistic. If the real structural behavior is obtained, the structures will be both more economical and safer. For this

purpose, nonlinear analysis methods are used very frequently today.

In the elastic analysis of structures, the load-displacement relationship of the bars is assumed to be linear-elastic. This assumption is valid only if there is no axial force on the bars. In fact, the normal force acting on the structural element changes the stiffness matrix of the element. Due to the change in the stiffness matrix of the element, the second-order effects make it necessary to write the equilibrium equations on the deformed system, not on the undeformed system.

If the load values on the structural system are not too large, it will be seen that the displacements obtained from the analysis are small in proportion to the dimensions of the structure. In this case, establishing the system equilibrium equations based on the undeformed position of the structure will not create large errors. However, if the normal force acting on the structural elements approaches the buckling load, the displacements increase greatly, and the error percentage reaches large levels. The change in the element stiffness, depending on the value of the axial force acting on it, is taken into account with dimensionless functions called Livesley stability functions [2].

In structural computational mechanics, the tangent stiffness matrix method is a linearization procedure that describes the stiffness of a system in response to small displacements applied to its current configuration. The tangent stiffness method is similar to the Newton-Raphson solution technique in solving nonlinear equations. The final solution is obtained as the sum of successive values calculated in each iteration cycle. Due to large displacements, as is known from nonlinear elasticity theory, there are nonlinear terms in the strain-displacement relations. These terms are taken into account while establishing the tangent stiffness matrix [3-5].

Geometrically nonlinear analysis is essential for accurately capturing the stability and ultimate response of structural systems subjected to large displacements and second-order effects. Crisfield [6] examined the nonlinear analysis of different types of structural elements in detail. In this study, the theoretical basis of geometrically nonlinear analysis was explained, and exact solutions of a simple system were obtained by the modified Newton-Raphson successive approximation method. Gu and Chan [7] developed nonlinear analysis approaches based on the tangent stiffness method and proposed a computational procedure integrated with stability functions. Rodrigues et al. [8] applied a combined tangent stiffness method to perform geometrically nonlinear analysis of space structural systems. The tangent stiffness matrix was calculated from the solution of the differential equation of equilibrium of a deformed infinitesimal element, considering axial load and shear deformation. The proposed combined tangent stiffness method provided an effective formulation for performing geometrically nonlinear analysis of space structures and for estimating the critical buckling load. Yang [9] evaluated the

stability behavior of steel structures under second-order effects and employed the tangent stiffness method in geometrically nonlinear analysis. Zhou et al. [10] developed a second-order inelastic analysis method for the design of box-section and H-section high-strength steel frames. In this study, second-order effects were taken into account using stability functions. To reduce the element degrees of freedom, the condensed tangent stiffness approach was adopted. Saka and Hayaloglu [11] and Saka and Kameshki [12] carried out the optimum design of steel frames using the stability functions defined by Livesley [2]. In these studies, the cross-sections of multi-storey, multi-span rigid frames were optimized by an iterative procedure to obtain the lightest structure. According to Saka and Kameshki [12], considering nonlinear behavior in the optimum design of such frames provides both greater savings and more realistic results. Saka and Ülker [13] carried out the optimum design under displacement and stress constraints using the tangent stiffness method in the geometrically nonlinear analysis of space frame systems. In this study, it was observed that some tension members transformed into compression members at the end of the nonlinear analysis. With the proposed optimum design algorithm, a realistic analysis was achieved, and a minimum-weight structure was obtained.

Mai et al. [14] developed a deep neural network (DNN)-based model integrated with the differential evolution (DE) algorithm for the optimal design of geometrically nonlinear space truss systems. In this study, the tangent stiffness matrix method was used to account for geometric nonlinearity. Joli et al. [15] investigated the geometric nonlinearity of cable structures with large displacements under geometric constraints. Geometric constraints were taken into account using penalty functions and tangent stiffness methods. Costa et al. [16] developed a formulation for the analysis of structural systems such as suspension bridges, cable-stayed bridges, and space trusses. Both geometric nonlinearity and material nonlinearity were included in the developed formulation. Damcı [17] examined the performance of structures under earthquake effects by considering their nonlinear behavior and presented a different static pushover method. In this study, geometrically nonlinear behavior was taken into account by considering stability functions. Although the Stability Functions and Tangent Stiffness methods are widely used for geometrically nonlinear structural analysis, this study implements both approaches as independent computational algorithms within the same computational environment and systematically evaluates their performance using the same benchmark problems.

The presented study demonstrates the effectiveness and accuracy of stability functions and the tangent stiffness matrix in the geometrically nonlinear analysis of structures. For validation, the results obtained from the nonlinear analysis of two frames and a space truss system with known exact solutions were compared. The results confirm that the

proposed approach yields consistent and reliable predictions in nonlinear structural analysis.

The novelty of this study lies in the development and implementation of two independent computational algorithms based on the Stability Functions and Tangent Stiffness methods for geometrically nonlinear structural analysis. Although both methods are well established in structural mechanics, this study presents their independent algorithmic implementations within the same computational environment and provides a systematic comparative evaluation using the same benchmark problems. The developed algorithms were coded in Visual Basic.NET and validated by comparing them with existing exact solutions in the literature and SAP2000 analyses. The results demonstrate the numerical accuracy, consistency, and reliability of both approaches for modeling geometrically nonlinear structural behavior under large displacements and second-order effects.

2. Nonlinear structural analysis

In linear analysis of structures, assuming that the structural deformations are very small, the element deformations are expressed as linear functions of the nodal displacements. The equilibrium equations are based on the undeformed geometry of the structure. Such an approximation is known as geometric linearity. The second approach in linear analysis is material linearity, which is represented by the linear elastic stress-strain relationship for the material. Linear analysis, also called first-order analysis, is generally sufficient to predict the performance of the most common types of engineering structures under service loading conditions. However, as the loads increase beyond the limits towards the failure range, the accuracy of linear analysis deteriorates. As the deformations increase, the structural material becomes increasingly nonlinear as it is stressed beyond the yield point. Due to its inherent limitations, linear analysis cannot be used to predict instability events and the elastic instability limit of structures. Recently, with the introduction of design codes based on the ultimate strength of structures, the use of nonlinear analysis in structural design has increased. In a nonlinear structural analysis, the load-deformation relations are nonlinear and are usually solved using iterative techniques.

2.1. Geometrically nonlinear analysis with stability functions

Geometrically nonlinear analysis with stability functions developed by Livesley [2] is used. In these formulations, the strain-displacement functions on the deformed geometry of the structure are expressed with a relation based on power series with six-digit accuracy. According to this relation, ϕ_1

is calculated first, and the other functions are found accordingly. The stability functions used are given below:

$$\phi_1 = \left(\frac{64 - 60\rho + 5\rho^2}{(16 - \rho)(4 - \rho)} \right) - \sum_{n=1}^7 \frac{a_n \rho^n}{2^{3n}} \quad (1)$$

$$\phi_2 = \left(\frac{\alpha^2}{3 - 3\phi_1} \right) \quad (2)$$

$$\phi_3 = 0.25(3\phi_2 + \phi_1) \quad (3)$$

$$\phi_4 = 0.5(3\phi_2 - \phi_1) \quad (4)$$

$$\phi_5 = \phi_1 \phi_2 \quad (5)$$

$$\alpha = 0.5\pi\sqrt{\rho} \quad (6)$$

In these equations, ρ is the ratio of the element axial load to the critical Euler buckling load. The values of the constant coefficients α_1 to α_7 are given in Table 1. In addition, the change of the stability functions in the range of $-4 \leq \rho \leq 4$ is evaluated in MS-EXCEL in two regions and shown in Table 2 and Table 3.

Fig. 1 illustrates the variation of the four stability functions used in this study. According to the sign convention adopted herein, a positive axial force represents tension, whereas a negative axial force represents compression. As the stability functions approach their singularity points, they exhibit asymptotic behavior because the denominators of the trigonometric expressions approach zero, representing the mathematical limits of the formulation. From a physical perspective, these functions describe the influence of axial force on the element stiffness and enable the incorporation of geometric stiffness effects into the element stiffness matrix.

Since nonlinear analysis methods are based on iterative procedures by nature, the analysis method developed in this study is also based on approximating a tolerance value for the axial force of the structural element. The stability functions developed by Livesley [2] in the form of force functions were used to perform the nonlinear analysis of the system with an iterative algorithm. It was observed that the analyses performed using stability functions provided accuracy in the solutions.

The basic principle in the analysis of geometrically nonlinear structures is to re-establish the equilibrium equations according to the deformed geometry. Therefore, the effect of large displacements in the structural system is taken into account in the method presented in this study. The stiffness matrix used in the analysis of the structural element, which has six degrees of freedom in a planar building system, is reconstructed at each step using recalculated stability functions [1].

Table 1. The values of constants in stability functions

α_1	α_2	α_3	α_4	α_5	α_6	α_7
1.5797363	0.1585859	0.027489	0.005475	0.001153	0.000249	5.45E-05

Table 2. Variation of stability functions in the range $-4 \leq \rho \leq 0$

ρ	1	2	3	4	5	6	7	SUM	ϕ_1
-4.00	-0.7899	0.3949	-0.1975	0.0987	-0.0494	0.0247	-0.0123	-0.5307	2.9307
-3.90	-0.7701	0.3754	-0.1830	0.0892	-0.0435	0.0212	-0.0103	-0.5211	2.9004
-3.80	-0.7504	0.3564	-0.1693	0.0804	-0.0382	0.0181	-0.0086	-0.5115	2.8697
-3.70	-0.7306	0.3379	-0.1563	0.0723	-0.0334	0.0155	-0.0072	-0.5018	2.8385
-3.60	-0.7109	0.3199	-0.1440	0.0648	-0.0292	0.0131	-0.0059	-0.4921	2.8068
-3.50	-0.6911	0.3024	-0.1323	0.0579	-0.0253	0.0111	-0.0048	-0.4823	2.7746
-3.40	-0.6714	0.2853	-0.1213	0.0515	-0.0219	0.0093	-0.0040	-0.4723	2.7418
-3.30	-0.6516	0.2688	-0.1109	0.0457	-0.0189	0.0078	-0.0032	-0.4623	2.7084
-3.20	-0.6319	0.2528	-0.1011	0.0404	-0.0162	0.0065	-0.0026	-0.4521	2.6743
-3.10	-0.6121	0.2372	-0.0919	0.0356	-0.0138	0.0053	-0.0021	-0.4418	2.6396
-3.00	-0.5924	0.2222	-0.0833	0.0312	-0.0117	0.0044	-0.0016	-0.4313	2.6042
-2.90	-0.5727	0.2076	-0.0753	0.0273	-0.0099	0.0036	-0.0013	-0.4206	2.5681
-2.80	-0.5529	0.1935	-0.0677	0.0237	-0.0083	0.0029	-0.0010	-0.4098	2.5312
-2.70	-0.5332	0.1799	-0.0607	0.0205	-0.0069	0.0023	-0.0008	-0.3988	2.4936
-2.60	-0.5134	0.1669	-0.0542	0.0176	-0.0057	0.0019	-0.0006	-0.3876	2.4551
-2.50	-0.4937	0.1543	-0.0482	0.0151	-0.0047	0.0015	-0.0005	-0.3762	2.4157
-2.40	-0.4739	0.1422	-0.0427	0.0128	-0.0038	0.0012	-0.0003	-0.3646	2.3755
-2.30	-0.4542	0.1306	-0.0375	0.0108	-0.0031	0.0009	-0.0003	-0.3528	2.3343
-2.20	-0.4344	0.1195	-0.0329	0.0090	-0.0025	0.0007	-0.0002	-0.3408	2.2922
-2.10	-0.4147	0.1089	-0.0286	0.0075	-0.0020	0.0005	-0.0001	-0.3285	2.2491
-2.00	-0.3949	0.0987	-0.0247	0.0062	-0.0015	0.0004	-0.0001	-0.3160	2.2049
-1.90	-0.3752	0.0891	-0.0212	0.0050	-0.0012	0.0003	-0.0001	-0.3032	2.1596
-1.80	-0.3554	0.0800	-0.0180	0.0040	-0.0009	0.0002	0.0000	-0.2902	2.1131
-1.70	-0.3357	0.0713	-0.0152	0.0032	-0.0007	0.0001	0.0000	-0.2769	2.0654
-1.60	-0.3159	0.0632	-0.0126	0.0025	-0.0005	0.0001	0.0000	-0.2633	2.0165
-1.50	-0.2962	0.0555	-0.0104	0.0020	-0.0004	0.0001	0.0000	-0.2494	1.9663
-1.40	-0.2765	0.0484	-0.0085	0.0015	-0.0003	0.0000	0.0000	-0.2353	1.9147
-1.30	-0.2567	0.0417	-0.0068	0.0011	-0.0002	0.0000	0.0000	-0.2208	1.8617
-1.10	-0.2172	0.0299	-0.0041	0.0006	-0.0001	0.0000	0.0000	-0.1910	1.7510
-1.00	-0.1975	0.0247	-0.0031	0.0004	0.0000	0.0000	0.0000	-0.1755	1.6932
-0.90	-0.1777	0.0200	-0.0022	0.0003	0.0000	0.0000	0.0000	-0.1597	1.6336
-0.80	-0.1580	0.0158	-0.0016	0.0002	0.0000	0.0000	0.0000	-0.1436	1.5722
-0.70	-0.1382	0.0121	-0.0011	0.0001	0.0000	0.0000	0.0000	-0.1271	1.5088
-0.60	-0.1185	0.0089	-0.0007	0.0000	0.0000	0.0000	0.0000	-0.1102	1.4434
-0.50	-0.0987	0.0062	-0.0004	0.0000	0.0000	0.0000	0.0000	-0.0929	1.3758
-0.40	-0.0790	0.0039	-0.0002	0.0000	0.0000	0.0000	0.0000	-0.0752	1.3058
-0.30	-0.0592	0.0022	-0.0001	0.0000	0.0000	0.0000	0.0000	-0.0571	1.2334
-0.20	-0.0395	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0385	1.1585
-0.10	-0.0197	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0195	1.0807

Table 3. Variation of stability functions in the range $0 \leq \rho \leq 4$

ρ	1	2	3	4	5	6	7	SUM	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	α
0.10	0.0197	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0200	0.9161	0.9808	0.9646	1.0131	0.8985	0.4967
0.20	0.0395	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0405	0.8289	0.9615	0.9283	1.0277	0.7970	0.7025
0.30	0.0592	0.0022	0.0001	0.0000	0.0000	0.0000	0.0000	0.0615	0.7381	0.9420	0.8910	1.0440	0.6953	0.8604
0.40	0.0790	0.0039	0.0002	0.0000	0.0000	0.0000	0.0000	0.0831	0.6434	0.9224	0.8527	1.0620	0.5935	0.9935
0.50	0.0987	0.0062	0.0004	0.0000	0.0000	0.0000	0.0000	0.1053	0.5445	0.9027	0.8132	1.0819	0.4915	1.1107
0.60	0.1185	0.0089	0.0007	0.0000	0.0000	0.0000	0.0000	0.1281	0.4411	0.8829	0.7724	1.1038	0.3894	1.2167
0.70	0.1382	0.0121	0.0011	0.0001	0.0000	0.0000	0.0000	0.1515	0.3328	0.8629	0.7303	1.1279	0.2871	1.3142
0.80	0.1580	0.0158	0.0016	0.0002	0.0000	0.0000	0.0000	0.1755	0.2192	0.8427	0.6868	1.1544	0.1847	1.4050
0.90	0.1777	0.0200	0.0022	0.0003	0.0000	0.0000	0.0000	0.2002	0.0999	0.8224	0.6418	1.1836	0.0822	1.4902
1.00	0.1975	0.0247	0.0031	0.0004	0.0000	0.0000	0.0000	0.2257	-0.0257	0.8019	0.5950	1.2157	-0.0206	1.5708
1.10	0.2172	0.0299	0.0041	0.0006	0.0001	0.0000	0.0000	0.2518	-0.1581	0.7812	0.5464	1.2509	-0.1235	1.6475
1.20	0.2370	0.0355	0.0053	0.0008	0.0001	0.0000	0.0000	0.2788	-0.2981	0.7603	0.4957	1.2895	-0.2266	1.7207
1.30	0.2567	0.0417	0.0068	0.0011	0.0002	0.0000	0.0000	0.3065	-0.4463	0.7392	0.4428	1.3320	-0.3300	1.7910
1.40	0.2765	0.0484	0.0085	0.0015	0.0003	0.0000	0.0000	0.3351	-0.6038	0.7180	0.3875	1.3788	-0.4335	1.8586
1.50	0.2962	0.0555	0.0104	0.0020	0.0004	0.0001	0.0000	0.3646	-0.7714	0.6964	0.3295	1.4304	-0.5373	1.9238
1.60	0.3159	0.0632	0.0126	0.0025	0.0005	0.0001	0.0000	0.3949	-0.9505	0.6747	0.2684	1.4873	-0.6413	1.9869
1.70	0.3357	0.0713	0.0152	0.0032	0.0007	0.0001	0.0000	0.4263	-1.1423	0.6527	0.2039	1.5501	-0.7455	2.0481
1.80	0.3554	0.0800	0.0180	0.0040	0.0009	0.0002	0.0000	0.4586	-1.3485	0.6304	0.1357	1.6198	-0.8501	2.1074
1.90	0.3752	0.0891	0.0212	0.0050	0.0012	0.0003	0.0001	0.4920	-1.5711	0.6078	0.0631	1.6972	-0.9549	2.1652
2.00	0.3949	0.0987	0.0247	0.0062	0.0015	0.0004	0.0001	0.5265	-1.8123	0.5849	-0.0144	1.7835	-1.0600	2.2214
2.10	0.4147	0.1089	0.0286	0.0075	0.0020	0.0005	0.0001	0.5622	-2.0749	0.5617	-0.0975	1.8800	-1.1655	2.2763
2.20	0.4344	0.1195	0.0329	0.0090	0.0025	0.0007	0.0002	0.5991	-2.3624	0.5381	-0.1870	1.9884	-1.2713	2.3299
2.30	0.4542	0.1306	0.0375	0.0108	0.0031	0.0009	0.0003	0.6373	-2.6790	0.5142	-0.2841	2.1108	-1.3775	2.3822
2.40	0.4739	0.1422	0.0427	0.0128	0.0038	0.0012	0.0003	0.6769	-3.0298	0.4898	-0.3901	2.2497	-1.4841	2.4335
2.50	0.4937	0.1543	0.0482	0.0151	0.0047	0.0015	0.0005	0.7179	-3.4216	0.4650	-0.5066	2.4083	-1.5911	2.4836
2.60	0.5134	0.1669	0.0542	0.0176	0.0057	0.0019	0.0006	0.7603	-3.8627	0.4398	-0.6358	2.5910	-1.6987	2.5328
2.70	0.5332	0.1799	0.0607	0.0205	0.0069	0.0023	0.0008	0.8044	-4.3642	0.4140	-0.7806	2.8031	-1.8067	2.5811
2.80	0.5529	0.1935	0.0677	0.0237	0.0083	0.0029	0.0010	0.8501	-4.9410	0.3876	-0.9445	3.0519	-1.9153	2.6284
2.90	0.5727	0.2076	0.0753	0.0273	0.0099	0.0036	0.0013	0.8975	-5.6130	0.3607	-1.1327	3.3475	-2.0245	2.6750
3.00	0.5924	0.2222	0.0833	0.0312	0.0117	0.0044	0.0016	0.9469	-6.4084	0.3331	-1.3523	3.7038	-2.1343	2.7207
3.10	0.6121	0.2372	0.0919	0.0356	0.0138	0.0053	0.0021	0.9981	-7.3676	0.3047	-1.6134	4.1409	-2.2449	2.7657
3.20	0.6319	0.2528	0.1011	0.0404	0.0162	0.0065	0.0026	1.0514	-8.5514	0.2755	-1.9312	4.6890	-2.3563	2.8099
3.30	0.6516	0.2688	0.1109	0.0457	0.0189	0.0078	0.0032	1.1069	-10.0552	0.2455	-2.3297	5.3959	-2.4686	2.8535
3.40	0.6714	0.2853	0.1213	0.0515	0.0219	0.0093	0.0040	1.1647	-12.0377	0.2145	-2.8486	6.3406	-2.5819	2.8964
3.50	0.6911	0.3024	0.1323	0.0579	0.0253	0.0111	0.0048	1.2249	-14.7849	0.1824	-3.5595	7.6660	-2.6963	2.9387
3.60	0.7109	0.3199	0.1440	0.0648	0.0292	0.0131	0.0059	1.2877	-18.8683	0.1490	-4.6053	9.6577	-2.8119	2.9804
3.70	0.7306	0.3379	0.1563	0.0723	0.0334	0.0155	0.0072	1.3532	-25.6214	0.1143	-6.3196	12.9822	-2.9288	3.0215
3.80	0.7504	0.3564	0.1693	0.0804	0.0382	0.0181	0.0086	1.4215	-39.0444	0.0780	-9.7026	19.6393	-3.0473	3.0620
3.90	0.7701	0.3754	0.1830	0.0892	0.0435	0.0212	0.0103	1.4928	-79.1375	0.0400	-19.7543	39.6288	-3.1676	3.1021

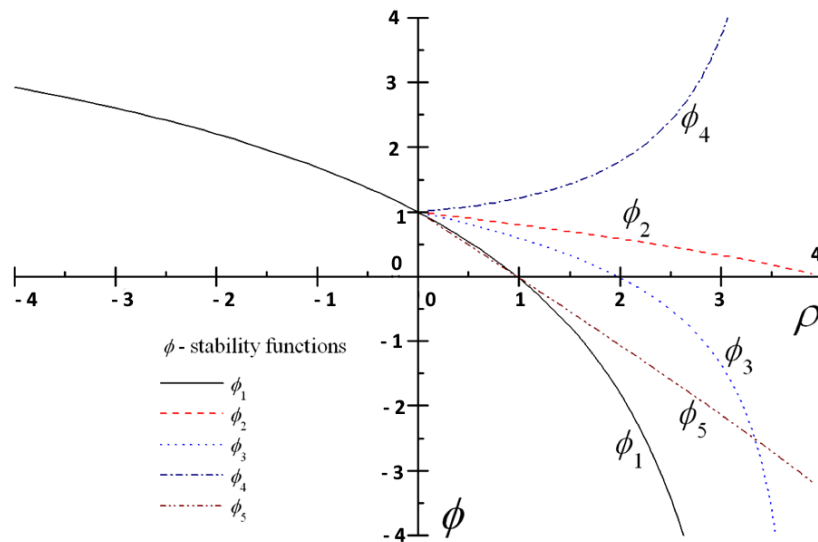


Fig. 1. Variation of stability functions with respect to the ρ

The stiffness matrix of a structural element with six degrees of freedom in local coordinates is given by

$$k = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI\phi_5}{L^3} & \frac{6EI\phi_2}{L^2} & 0 & -\frac{12EI\phi_5}{L^3} & \frac{6EI\phi_2}{L^2} \\ 0 & \frac{6EI\phi_2}{L^2} & \frac{4EI\phi_3}{L} & 0 & -\frac{6EI\phi_2}{L^2} & \frac{2EI\phi_4}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI\phi_5}{L^3} & \frac{6EI\phi_2}{L^2} & 0 & \frac{12EI\phi_5}{L^3} & -\frac{6EI\phi_2}{L^2} \\ 0 & \frac{6EI\phi_2}{L^2} & \frac{2EI\phi_4}{L} & 0 & -\frac{6EI\phi_2}{L^2} & \frac{4EI\phi_3}{L} \end{bmatrix} \quad (7)$$

where k is the geometrically nonlinear stiffness matrix of the member; E is the modulus of elasticity; A is the cross-sectional area; I is the moment of inertia; L is the member length; and ϕ_2 , ϕ_3 , ϕ_4 , and ϕ_5 are the Livesley stability functions.

In this study, a new algorithm has been developed for the analysis of geometrically nonlinear structural systems with stability functions. The algorithm is coded in the Visual Basic .NET programming language on the MS Visual Studio 2019 platform [18, 19]. The stability functions flow diagram is given in Fig. 2. This iteration is continued until the difference between the axial loads of the elements used in the calculation of the stability functions and the loads obtained at the end of the same cycle is within a certain tolerance.

In the developed VB.NET algorithms, the solution was obtained using an iterative procedure based on a displacement convergence criterion. For all analyses, the convergence tolerance was set to 0.001, and the maximum number of iterations was limited to 100. The iterative solution process was terminated when either the displacement convergence criterion was satisfied or the maximum number of iterations was reached. In this study, the external load was applied in a

single load step. For the SAP2000 analyses, the default convergence settings of the software were adopted.

2.2. Geometrically nonlinear analysis with tangent stiffness method

When large displacements occur in a structure, the force balance equations must be formulated for the deformed configuration of the structure. This means that the linear relation $P = KU$ between the forces P applied to the structure and the displacements U can no longer be used in nonlinear analysis. To account for the effects of changes in geometry as the applied loading increases, a step-by-step linearization process is applied to the nonlinear problem, and solutions are obtained for the displacements U . Each linear step represents a residual load vector.

Due to the occurrence of large displacements, the strain-displacement equations contain nonlinear terms. These nonlinear terms are included in the matrix when constructing the K_T system stiffness matrix [5, 13]. In nonlinear terms in the strain-displacement equations, the k_T element stiffness matrix is expressed as

$$k_T = k_E + k_G \quad (8)$$

where, k_E is the standard element elastic stiffness matrix and k_G is the matrix called the element geometric stiffness matrix. The elastic and geometric stiffness matrices are calculated for each element and then combined into the total stiffness matrix K_T is expressed as

$$K_T = K_E + K_G \quad (9)$$

where, K_T , K_E and K_G are the matrices called tangent stiffness, elastic stiffness, and geometric stiffness matrices of the entire system, respectively. The K_T matrix is related to the changes in the geometry of the structure and is obtained by combining the element elastic and geometric stiffness matrices. The geometric stiffness matrix is a function of the bar forces. Since the bar force is not present at the beginning of the iteration, K_G becomes zero, which turns the first step into a linear analysis. The bar forces become available only after the first step of the iteration, and the geometric stiffness matrix is established. Thus, the established K_G geometric stiffness matrix is used to obtain the K_T tangent stiffness matrix.

In this study, the tangent stiffness method algorithm [13] is taken as a basis and coded in the Visual Basic .NET programming language. The steps of the method are given below:

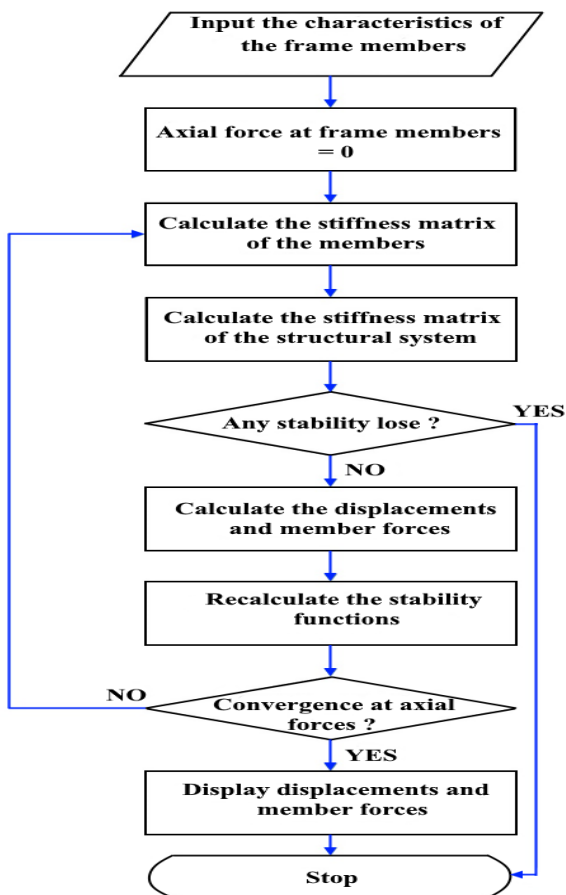


Fig. 2. Geometrically nonlinear analysis flowchart using stability functions

1. Perform linear analysis of the structure and calculate the joint displacements.
2. Add the calculated displacements to the previous coordinates to calculate the new joint coordinates.
3. Calculate the bar forces with the new bar lengths calculated in the second step.
4. Calculate the balancing forces from the total balance of the bar force components calculated in the third step for each joint.
5. Calculate the compensating load vector with the balancing forces calculated in the fourth step.
6. Construct the K_T tangent stiffness matrix and calculate its determinant. If the determinant is negative, system stability has failed; stop the process. If the determinant is positive, analyze the deformed structure under the compensatory load vector and calculate the displacement increments.
7. Add the displacement increments calculated in the sixth step to the joint coordinates calculated in the second step to obtain the new joint coordinates.
8. Repeat steps 3-7 until sufficient precision is achieved in the compensation load vector.

3. Nonlinear structural analysis with SAP2000

Finite element-based SAP2000 software is widely used in structural analysis [21]. Different structural systems were analyzed with verification examples in the SAP2000 program, and its accuracy was demonstrated. Almost 100% of the results were obtained in the verification examples. To demonstrate the validity of the algorithm developed, examples solved with stability functions and the tangent stiffness method were also solved with the SAP2000 program. There are two options to obtain geometrically nonlinear analysis results in the program:

First, open Define/Load Case; the analysis type is selected as Nonlinear, and the geometric nonlinearity parameter is “P-Delta plus Large Displacements”. The settings made for this are given in Fig. 3. When the settings in Fig. 3 are made in SAP2000, the nonlinear bar forces and displacements are obtained correctly.

In the second option, nonlinear static analysis is selected as displacement-controlled in the “Other Parameters” in the Load Case Data window, and pushover analysis is performed. However, in the pushover analysis performed, it was observed that the displacement at the top point was taken as 4% of the structure height by default. This limitation is a limit value recommended in ATC-40, FEMA 356, FEMA 440, and ASCE 41-13 NSP Specifications [22-25] to prevent the collapse of the structure and to ensure the life safety performance level. Based on this limit, nonlinear bar forces and displacements cannot be obtained at their true value.

If it is desired to obtain nonlinear results with their real values in pushover analysis, the maximum displacement at

the top of the structure obtained from the first option is taken into account. This maximum displacement is entered as the displacement upper limit in “Other Parameters”. Thus, the displacement-controlled pushover analysis is performed step by step. Nonlinear bar forces and displacements are obtained with their real values, as in the first option.

In the displacement-controlled pushover analysis, it is seen in Fig. 4 that the displacement-shear force graph is the same according to FEMA 356 [23] and ASCE 41-13 NSP [25] Specifications (ASCE 41-13 NSP graph is given as an

example). This graph is similar to the Tangent Stiffness method graph [13]. To maintain structural stability and ensure that the determinant of the stiffness matrix remains non-negative, the analysis should be performed in a displacement-controlled manner up to an appropriate displacement level. Following the description of the SAP2000 implementation, the developed algorithms were validated through numerical examples, and the obtained results were compared with the benchmark solutions available in the literature and the SAP2000 analysis results.

S Load Case Data - Nonlinear Static

Load Case Name: Set Def Name Modify/Show...

Initial Conditions:
☒ Zero Initial Conditions - Start from Unstressed State
☐ Continue from State at End of Nonlinear Case
 Important Note: Loads from this previous case are included in the current case

Modal Load Case:
 All Modal Loads Applied Use Modes from Case:

Loads Applied:

Load Pattern	Load Name	Scale
G	G	1.

Add Modify Delete

Other Parameters:
 Load Application: Modify/Show...
 Results Saved: Modify/Show...
 Nonlinear Parameters: Modify/Show...

Load Case Type: Design...

Analysis Type:
☐ Linear
☒ Nonlinear

Geometric Nonlinearity Parameters:
☐ None
☐ P-Delta
☒ P-Delta plus Large Displacements

Mass Source:

OK Cancel

Fig. 3. Geometric nonlinearity options in SAP2000

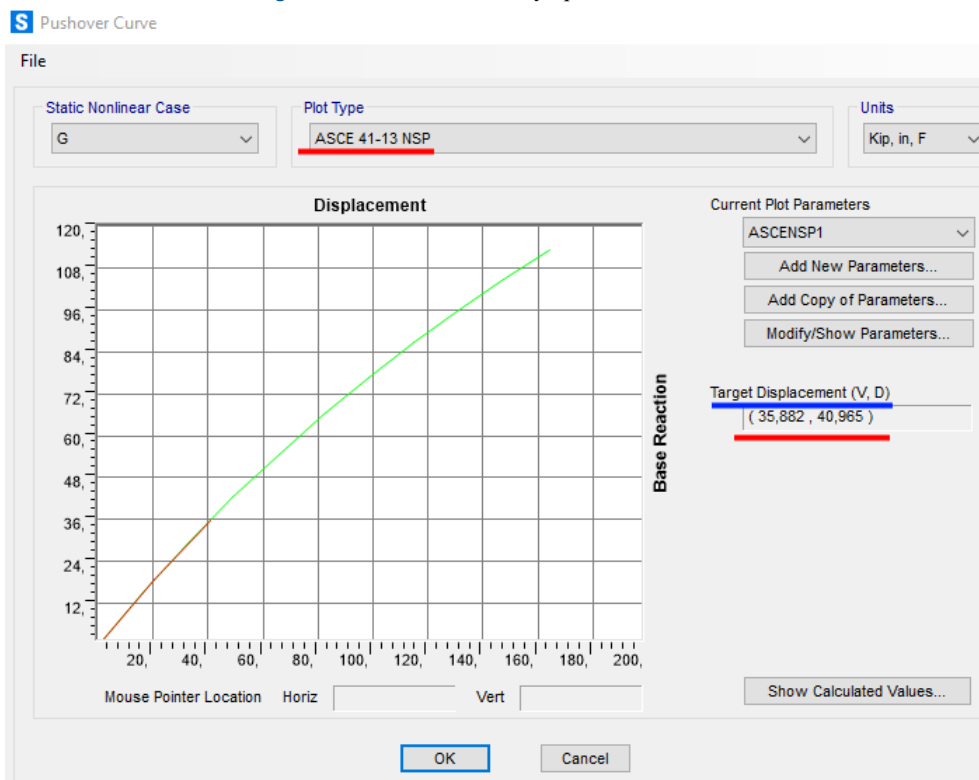


Fig. 4. Displacement controlled pushover analysis in SAP2000

4. Numerical examples

In this study, the analyses of two planar frames and a space truss system were performed on the geometrically nonlinear behavior of the structures with the stability functions and tangent stiffness method algorithms presented. The calculations were performed with the algorithm coded in the Visual Basic .NET programming language on the MS Visual Studio 2019 platform. In order to demonstrate the validity of the developed algorithms, the same examples were solved with the SAP2000 program, and the results obtained were compared.

4.1. Simple frame system

To compare the nonlinear analysis results with stability functions and the tangent stiffness method, a single-story and single-span steel frame analysis was performed [26]. The geometry of the frame is shown in Fig. 5. The elastic modulus $E = 20693 \text{ kN/cm}^2$, moments of inertia, and cross-sectional areas are taken as $I = 4162.31 \text{ cm}^4$ and $A = 64.516 \text{ cm}^2$, being the same for all bars. As seen in Fig. 5, the horizontal load at point B is taken as one percent of the vertical loads at points C and D, and the loads are increased starting from zero. The system is analyzed using stability functions for each load case, and the horizontal displacement of point B is compared with the tangent stiffness method solutions. Comparative results are shown in Table 4 and Fig. 6. In addition, linear analysis results are also given, and the increase in the normal force carried by the EF bar is shown. As seen in Table 5, as the load increases, the horizontal displacement and normal force also increase, and when the vertical loads reach the value of 259.1 kN, the system loses its stability.

As shown in Fig. 6, the nonlinear response exhibits a pronounced reduction in structural stiffness as the applied load increases. This behavior indicates that the structural

system is approaching the critical load condition and that the tangent stiffness decreases progressively. The rapid increase in displacement observed during the final loading stages is a characteristic feature of nonlinear stability behavior, indicating that the equilibrium path is approaching the limit point.

The geometrically nonlinear analysis revealed a redistribution of the bending moment diagrams compared with the linear analysis. Fig. 7 presents a comparison of the bending moment diagrams obtained from the linear and geometrically nonlinear analyses. The observed differences clearly demonstrate the influence of second-order effects on the distribution of internal forces.

4.2. 56-bar space truss system

In order to prove the validity of the Tangent Stiffness Method algorithm developed in this study, the analysis of the 56-bar space truss system [20] was performed. The geometrically nonlinear analysis of this system was also examined with the SAP2000 program.

4.2.1. Nonlinear analysis of 56-bar space truss system using tangent stiffness method

In order to compare the results of the geometrically nonlinear analysis, the data units [20] were taken as they are. The cross-sectional area of all bars of the 56-bar space truss system given in Fig. 8 is $A = 4 \text{ in}^2$, and the elasticity modulus is $E = 1000 \text{ ksi}$. The dimensions and loading conditions of the space truss system are also shown in the same figure. As can be seen from these figures, the external loads were taken as 20 kips in the negative Z-axis direction at nodes 1, 2, 3 and 4, and 10 kips in the positive X-axis direction at nodes 1, 3, 5, 7, 9, 11, 13 and 15.

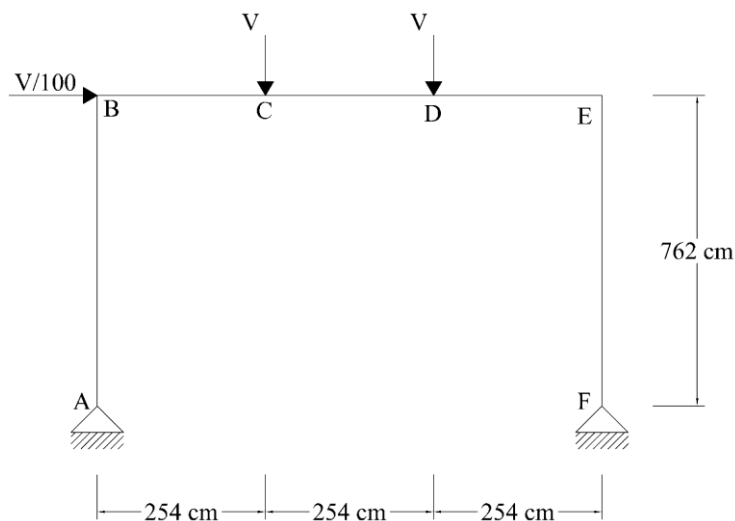


Fig. 5. Geometry and loading of simple frame

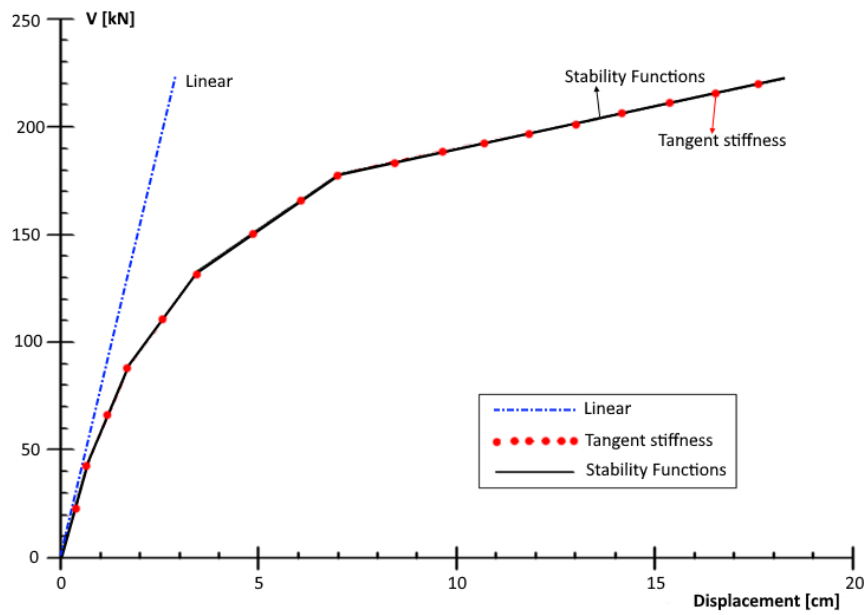


Fig. 6. Displacements of simple frame

Table 4. Comparative results of the simple frame

V (kN)	Horizontal Displacement of Joint B (cm)			
	Linear Analysis	Stability Function	Tangent Stiffness Method	SAP2000
44.50	0.570	0.700	0.708	0.71
89.00	1.158	1.730	1.740	1.745
133.50	1.720	3.470	3.480	3.481
178.00	2.300	7.020	7.080	7.098
222.50	2.870	18.280	18.196	18.198
44.50	0.570	0.700	0.708	0.71

Table 5. Variation of horizontal displacement and normal force in simple frame

V (kN)	Horizontal Displacement of Joint B (cm)	Normal Force of EF Bar (kN)	Tangent Stiffness Method	SAP2000
44.50	0.690	45.000	45.020	45.150
89.00	1.730	90.000	90.270	90.170
133.50	3.470	136.000	135.980	136.030
178.00	7.020	183.000	182.810	182.580
222.50	18.280	235.000	234.960	235.100
231.40	24.310	248.000	248.060	248.026
240.30	35.000	265.000	265.110	265.240
249.20	59.310	290.000	290.650	290.330
258.10	182.200	385.000	384.940	385.080
259.00	267.400	443.000	443.320	443.400
259.10	Loss of Stability	Loss of Stability	Loss of Stability	Loss of Stability

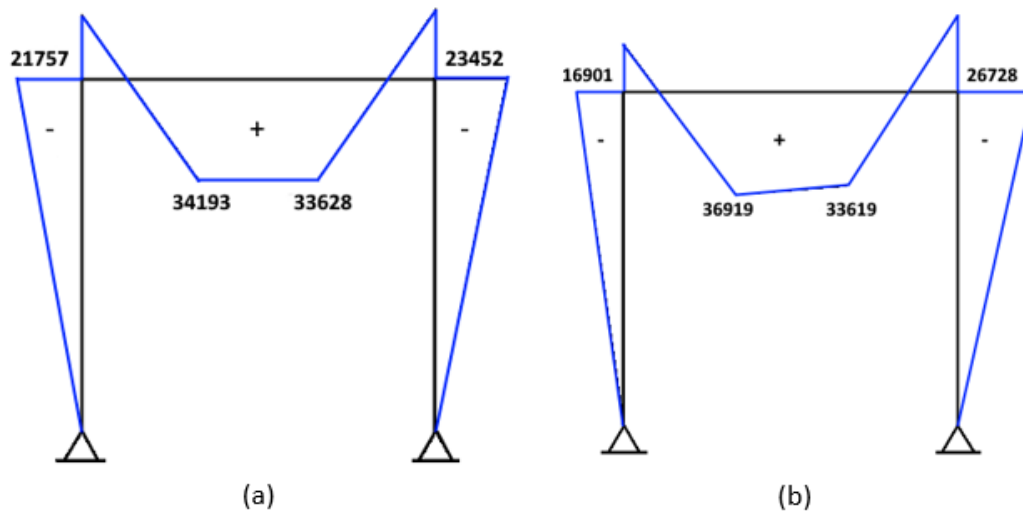


Fig. 7. Comparison of bending moment diagrams (kN·cm) for the simple frame: (a) Linear analysis; (b) Geometrically nonlinear analysis using the Stability Functions method

The results obtained from the analysis of the 56-bar space truss system with the Tangent Stiffness Method algorithm are given in Table 6 and Table 7. As can be seen from the analysis results in the tables, large changes occurred in the displacements and bar forces in the nonlinear analysis. For

example, in Table 7, the forces in bar no. 53 were -7.546 kips compressive force in the linear analysis, while they became 11.460 kips with a change of 252% in the nonlinear analysis. The nonlinear analysis was obtained in 6 steps with the tangent stiffness method and the results were obtained [20].

Table 6. Linear and nonlinear displacements of 56-bar space truss system calculated by tangent stiffness method

Linear analysis joint displacements (in)				Nonlinear analysis joint displacements (in)			
Joints	X	Y	Z	Joints	X	Y	Z
1	116.80	-14.96	8.00	1	161.87	-19.47	-1.48
2	116.89	-14.99	-16.64	2	158.57	-20.90	-37.14
3	117.23	-14.88	8.96	3	164.09	-19.41	-0.55
4	116.86	-14.90	-14.40	4	160.22	-20.83	-34.83
5	83.19	-12.81	7.95	5	111.91	-16.84	2.98
6	82.88	-12.93	-15.44	6	108.44	-18.31	-30.63
7	82.62	-12.67	10.16	7	113.30	-16.74	5.99
8	82.76	-12.79	-12.85	8	110.31	-18.23	-27.33
9	48.55	-8.70	7.90	9	63.05	-10.26	7.19
10	48.72	-8.53	-12.13	10	60.94	-11.27	-20.97
11	49.17	-8.53	8.67	11	65.11	-10.13	7.88
12	48.88	-8.37	-11.30	12	62.62	-11.23	-20.20
13	19.45	-10.67	3.95	13	22.88	-13.70	4.55
14	19.16	-10.83	-8.83	14	21.68	-16.17	-12.92
15	18.84	-10.51	7.17	15	24.53	-13.64	8.19
16	10.00	-10.67	-5.65	16	23.72	-16.31	-9.32

Table 7. Linear and nonlinear bar forces calculated by the tangent stiffness method of a 56-bar space truss system

Linear Analysis											
Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)
1	-6.459	2	3.541	3	3.541	4	3.541	5	3.570	6	3.570
7	-6.430	8	3.570	9	-6.790	10	3.210	11	3.210	12	3.210
13	1.855	14	1.855	15	-8.145	16	1.855	17	65.858	18	-147.118
19	65.858	20	-55.122	21	0.793	22	-55.122	23	0.793	24	-20.000
25	94.142	26	119.549	27	-94.142	28	24.878	29	-25.874	30	24.878
31	25.874	32	-20.000	33	-5.007	34	7.089	35	7.188	36	-5.048
37	-4.539	38	5.959	39	1.333	40	-2.623	41	59.121	42	-56.710
43	25.623	44	-21.329	45	-74.213	46	43.290	47	-41.044	48	12.004
49	-7.546	50	-6.710	51	-7.711	52	-4.662	53	-7.546	54	-6.710
55	-7.711	56	-4.662								
Nonlinear Analysis											
Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)	Member Number	Force (Kips)
1	-7.340	2	-2.100	3	1.420	4	2.380	5	2.370	6	1.260
7	-6.280	8	3.160	9	-6.540	10	2.630	11	2.960	12	2.960
13	5.970	14	1.960	15	-12.250	16	1.690	17	100.050	18	-188.660
19	99.300	20	-78.340	21	14.970	22	-78.410	23	13.110	24	-19.590
25	-125.530	26	162.880	27	-125.810	28	52.030	29	-36.500	30	50.510
31	-36.500	32	-17.710	33	-13.510	34	10.650	35	7.590	36	-7.350
37	-4.670	38	5.670	39	1.130	40	-2.510	41	55.150	42	-69.250
43	38.580	44	-34.300	45	-90.930	46	51.720	47	-53.400	48	26.230
49	-2.720	50	-4.540	51	-7.520	52	-4.010	53	11.460	54	0.880
55	-4.410	56	-4.880								

4.2.2. Nonlinear analysis of 56-bar space truss system in SAP2000 program

It was explained in detail how to perform a geometrically nonlinear analysis with the SAP2000 program. The analysis of the 56-bar space truss system was first performed by selecting the analysis type Nonlinear and the geometric nonlinearity parameter “P-Delta plus Large Displacements”.

The results obtained from the nonlinear analysis performed with the SAP2000 program; joint displacements are given in Table 8, and bar forces are given in Table 9. It was observed that these results were the same as the results of the Tangent Stiffness Method.

In the second option of performing geometrically nonlinear analysis with the SAP2000 program, the pushover analysis method is used. The analysis is displacement-controlled. The height of the space truss system is 960 inches, and by default, $960 \times 0.04 = 38.4$ inches of displacement is allowed at the top point. When the top point displacement limitation is taken as 38.4 inches, the bar forces and

displacements are not obtained with their real values. In order to reach the real results of the geometrically nonlinear analysis, the displacement value of joint 3 in Table 8, $U_1=164.08$ inches, is taken as the limit displacement. Displacement-controlled pushover analysis was carried out step by step with the nonlinear analysis settings shown in Fig. 9. The nonlinear bar forces and displacements are obtained with their real values as in Table 8 and Table 9 in the first option.

Displacement-controlled pushover analysis was completed in 10 steps, and the top point displacement was calculated as 164.08 inches. The pushover curve obtained according to ASCE 41-13 NSP Specification is shown in Fig. 10. The performance point was determined as (35.882; 40.965). In order to prevent the collapse of the structure and to ensure the life safety performance level, the upper displacement limit should be taken into consideration not as 164.08 inches but as 40.965 inches recommended by ASCE 41-13 NSP Specification.

Table 10 summarizes the nonlinear static analysis parameters used for the 56-bar space truss system example. The table presents the essential modeling and solution settings required for the numerical analysis and serves as a complement to the software screenshots provided in Figs. 3, 4, 9, and 10.

Table 11 summarizes the comparative validation results of the proposed nonlinear solution algorithms. The displacement and member force results presented in detail in Tables 4–9 were evaluated in terms of the maximum relative errors with respect to the benchmark solutions available in the literature

and the SAP2000 analysis results. For the simple frame example, the maximum relative error was found to be 1.408% for the nodal displacements and 0.332% for the axial forces in member EF. Similarly, for the 56-bar space truss system, the maximum relative error was calculated as 0.139% for the nodal displacements and 0.604% for the member forces. These low relative error values demonstrate excellent agreement between the proposed algorithms, the benchmark solutions available in the literature, and the SAP2000 analysis results, thereby confirming the accuracy and reliability of the developed computational procedures.

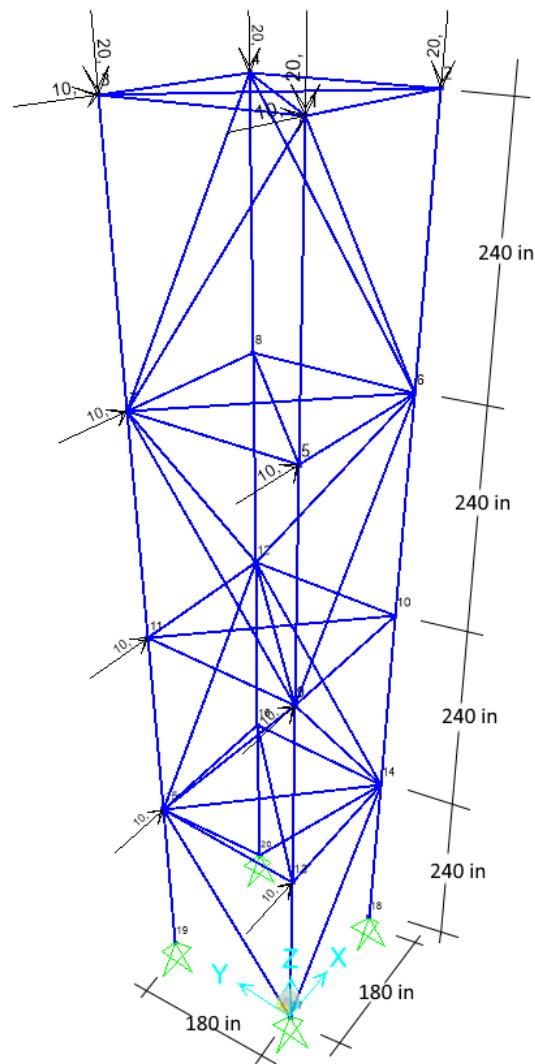


Fig. 8. Geometry and loading of the 56-bar space truss system

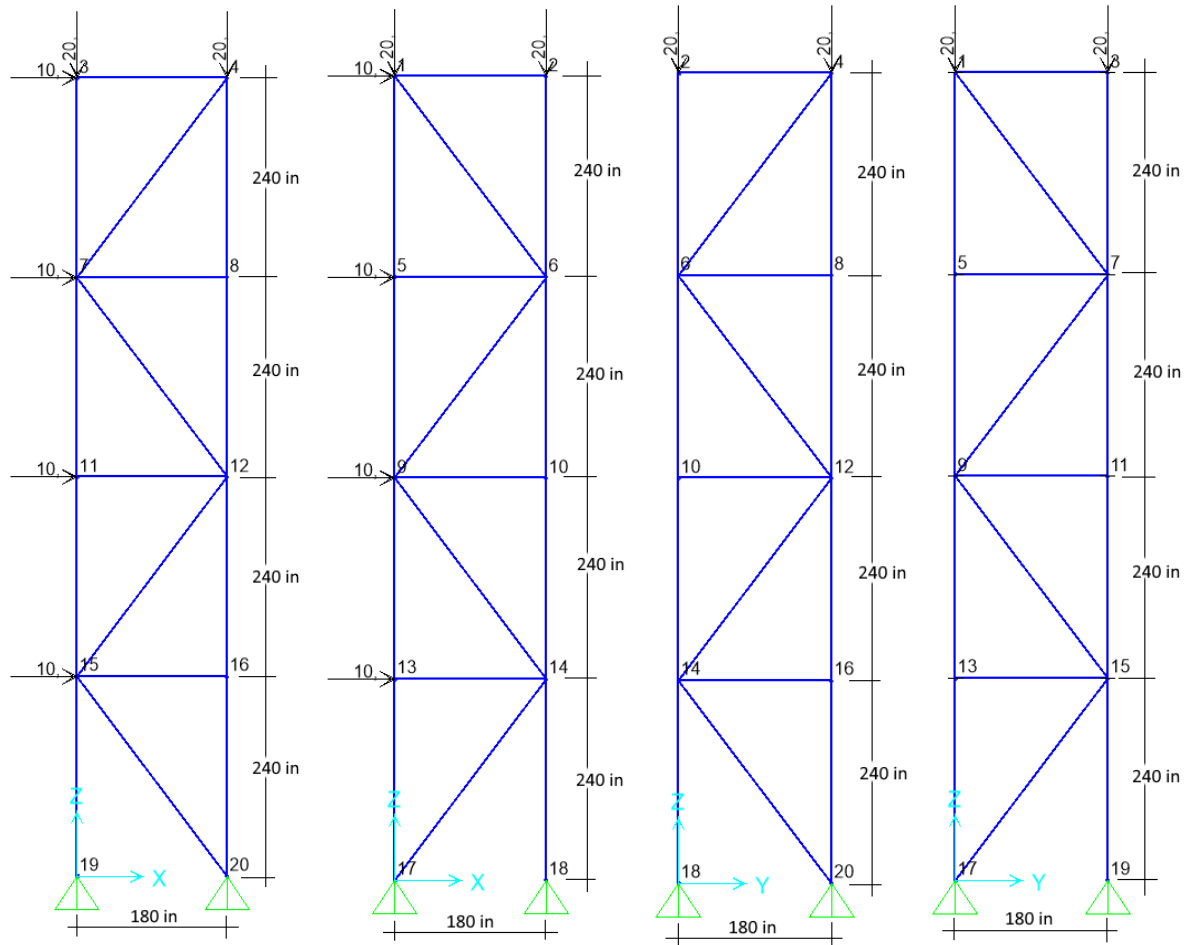


Fig. 8. Continued

Table 8. Nonlinear displacements of 56-bar space truss system calculated with SAP2000 P-Delta plus large displacements option

Joint	Output Case	Case Type	Step Type	U1 (in)	U2 (in)	U3 (in)
1	G	NonStatic	Max	161.86	-19.47	-1.48
2	G	NonStatic	Max	158.56	-20.90	-37.14
3	G	NonStatic	Max	164.08	-19.41	-0.55
4	G	NonStatic	Max	160.22	-20.83	-34.83
5	G	NonStatic	Max	111.91	-16.84	2.98
6	G	NonStatic	Max	108.44	-18.30	-30.63
7	G	NonStatic	Max	113.29	-16.74	5.99
8	G	NonStatic	Max	110.31	-18.23	-27.33
9	G	NonStatic	Max	63.05	-10.25	7.18
10	G	NonStatic	Max	60.93	-11.27	-20.97
11	G	NonStatic	Max	65.11	-10.13	7.88
12	G	NonStatic	Max	62.61	-11.23	-20.20
13	G	NonStatic	Max	22.88	-13.70	4.55
14	G	NonStatic	Max	21.68	-16.17	-12.92
15	G	NonStatic	Max	24.53	-13.64	8.19
16	G	NonStatic	Max	23.72	-16.31	-9.32
17	G	NonStatic	Max	0	0	0
18	G	NonStatic	Max	0	0	0
19	G	NonStatic	Max	0	0	0
20	G	NonStatic	Max	0	0	0
1	G	NonStatic	Max	161.86	-19.47	-1.48

Table 9. Nonlinear bar forces calculated with SAP2000 P-Delta plus large displacements option of the 56-bar space truss system

Frame	Station (in)	Output Case	Case Type	Step Type	P (Kip)	Frame	Station (in)	Output Case	Case Type	Step Type	P (Kip)
1	180	G	NonStatic	Max	-7.344	29	240	G	NonStatic	Max	-36.495
2	180	G	NonStatic	Max	-2.098	30	240	G	NonStatic	Max	50.511
3	180	G	NonStatic	Max	1.419	31	240	G	NonStatic	Max	-36.495
4	180	G	NonStatic	Max	2.380	32	240	G	NonStatic	Max	-17.713
5	180	G	NonStatic	Max	2.369	33	254.56	G	NonStatic	Max	-13.505
6	180	G	NonStatic	Max	1.255	34	254.56	G	NonStatic	Max	10.645
7	180	G	NonStatic	Max	-6.284	35	254.56	G	NonStatic	Max	7.587
8	180	G	NonStatic	Max	3.157	36	254.56	G	NonStatic	Max	-7.354
9	180	G	NonStatic	Max	-6.543	37	254.56	G	NonStatic	Max	-4.672
10	180	G	NonStatic	Max	2.632	38	254.56	G	NonStatic	Max	5.667
11	180	G	NonStatic	Max	2.963	39	254.56	G	NonStatic	Max	1.133
12	180	G	NonStatic	Max	2.963	40	254.56	G	NonStatic	Max	-2.508
13	180	G	NonStatic	Max	5.969	41	300	G	NonStatic	Max	55.151
14	180	G	NonStatic	Max	1.955	42	300	G	NonStatic	Max	-69.250
15	180	G	NonStatic	Max	-12.250	43	300	G	NonStatic	Max	38.576
16	180	G	NonStatic	Max	1.688	44	300	G	NonStatic	Max	-34.295
17	240	G	NonStatic	Max	100.048	45	300	G	NonStatic	Max	-90.927
18	240	G	NonStatic	Max	-188.658	46	300	G	NonStatic	Max	51.716
19	240	G	NonStatic	Max	99.300	47	300	G	NonStatic	Max	-53.394
20	240	G	NonStatic	Max	-78.337	48	300	G	NonStatic	Max	26.234
21	240	G	NonStatic	Max	14.972	49	300	G	NonStatic	Max	-2.724
22	240	G	NonStatic	Max	-78.413	50	300	G	NonStatic	Max	-4.542
23	240	G	NonStatic	Max	13.110	51	300	G	NonStatic	Max	-7.518
24	240	G	NonStatic	Max	-19.592	52	300	G	NonStatic	Max	-4.009
25	240	G	NonStatic	Max	-125.529	53	300	G	NonStatic	Max	11.454
26	240	G	NonStatic	Max	162.878	54	300	G	NonStatic	Max	0.884
27	240	G	NonStatic	Max	-125.808	55	300	G	NonStatic	Max	-4.409
28	240	G	NonStatic	Max	52.029	56	300	G	NonStatic	Max	-4.880

Table 10. Nonlinear static analysis settings for the 56-bar space truss

Parameter	Value
Analysis Type	Nonlinear Static
Initial Conditions	Zero Initial Conditions (Start From Unstressed State)
Modal Load Case	MODAL
Geometric Nonlinearity	P-Delta Plus Large Displacements
Mass Source	Previous
Load Application Method	Displacement Control
Control Displacement	Monitored Displacement
Target Displacement	40.9655 in
Pushover Load Pattern	G
Pushover Plot Type	ASCE 41-13 NSP
Unit System	kip, in, °F

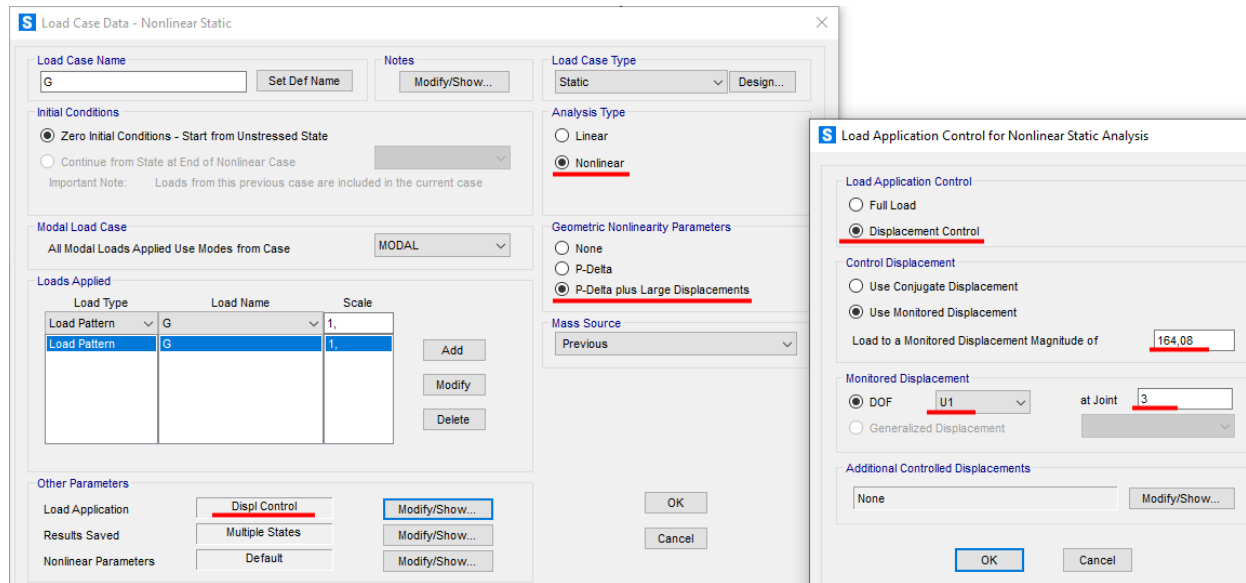


Fig. 9. SAP2000 pushover analysis settings of 56-bar space truss system

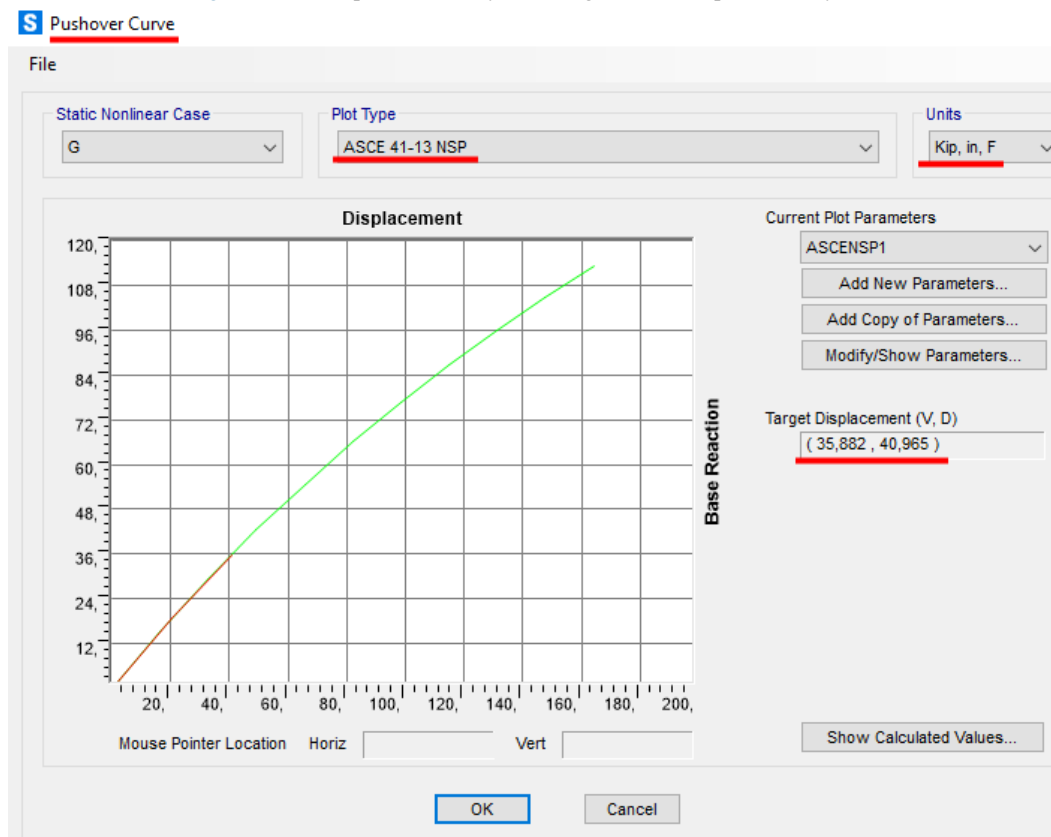


Fig. 10. SAP2000 pushover analysis curve according to ASCE 41-13 NSP Specification

Table 11. Summary of the validation results for the proposed geometrically nonlinear analysis method

Structural Example	Compared Response	Reference Solution	Number of Values	Max. Relative Error (%)
Simple Frame	Horizontal Displacement of Joint B	Tangent Stiffness Method	5	1.130
Simple Frame	Horizontal Displacement of Joint B	SAP2000	5	1.408
Simple Frame	Normal Force of EF Bar	Tangent Stiffness Method	10	0.299
Simple Frame	Normal Force of EF Bar	SAP2000	10	0.332
56-Bar Space Truss	Nonlinear Joint Displacements	SAP2000	48	0.139
56-Bar Space Truss	Nonlinear Member Forces	SAP2000	56	0.604

4.3. Analysis of 3-Span 10-story plane frame

The analysis and optimum design of the 3-span 10-story steel plane frame shown in Fig. 11 were performed by Saka and Kameshki [12] using stability functions. The dimensions of the frame, the applied loading system, and the grouping of the elements are shown in Fig. 11. The columns are grouped under groups 1-8, and all the beams are grouped under group 9. The yield stress in the material is taken as 230 N/mm², and the elasticity modulus is 210 kN/mm². The lateral displacement of the top floor is limited to 118.3 mm.

In the optimum design, a relation is required that connects the cross-sectional area to the moment of inertia and the section modulus. These relations are given as follows [12]:

$$I_x = a_1 A^{b_1}, \quad I_y = a_2 A^{b_2}, \quad Z = a_3 A^{b_3} \quad (10)$$

where A is the cross-sectional area of the bar, I_x and I_y are the moments of inertia of the section about the x-x and y-y axes, and Z is the section modulus. The coefficients a_{1-3} and b_{1-3} in Eq. (10) are calculated as $a_1 = 1.3162$, $b_1 = 2.165$, $a_2 = 0.598$, $b_2 = 1.822$, $a_3 = 1.016$, $b_3 = 1.571$ for wide-flange W sections. Analysis and design of the 10-story plane frame [12] was started by selecting the cross-sectional area of all bars as $A=50 \text{ cm}^2$ in the first step of the iteration, and the optimum cross-sectional areas were obtained at the end of 5 steps as seen in Table 12. The moment of inertia and section modulus calculated with Eq. (10) are also given in Table 12.

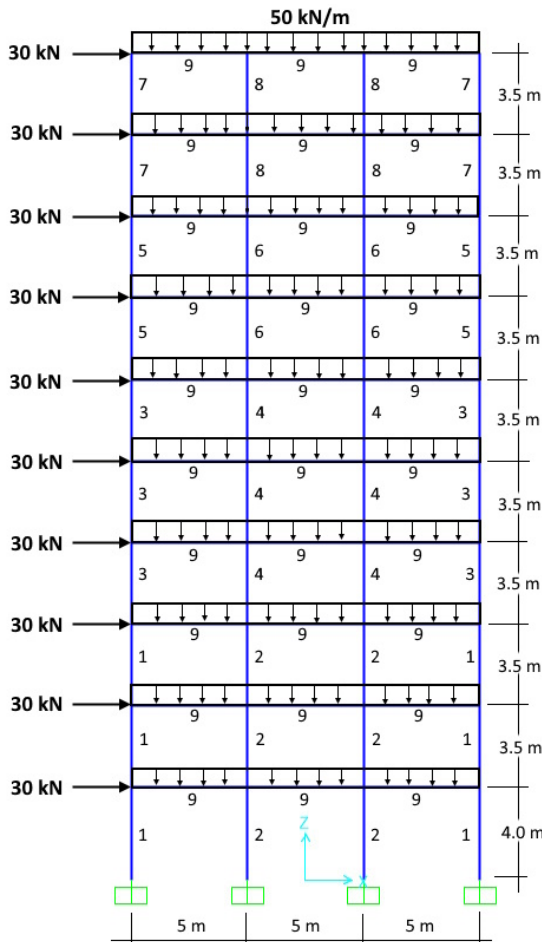


Fig. 11. 3-span 10-storey plane frame

Table 12. Section properties of 10-storey plane frame calculated from optimum design

Group Number	Section Area (cm ²) A	Moment of Inertia (cm ⁴) I _x	Moment of Inertia (cm ⁴) I _y	Section Modulus (cm ³) Z
1	176.30	96041	7402	3434
2	288.20	278330	18124	7431
3	125.10	45697	3962	2003
4	176.60	96396	7425	3443
5	84.77	19677	1950	1087
6	111.50	35618	3212	1672
7	57.77	8578	969	595
8	61.77	9916	1095	661
9	110.10	34657	3139	1639

Nonlinear analysis and design of the same frame with stability functions were carried out under the restrictions given by Saka and Kameshki [12], and the horizontal displacement of the top floor was calculated as 5.21 cm with the optimum cross-sectional areas obtained.

Geometrically nonlinear analysis of the 10-storey frame was performed with the Stability Functions and Tangent Stiffness Method Algorithms developed in this study. The section properties in Table 12 were used in these analyses. Horizontal displacement of the top point was calculated as 5.19 cm with the stability functions algorithm and 5.218 cm with the Tangent Stiffness Method Algorithms. These results were found to be the same as the results of Saka and Kameshki [12].

In the geometrically nonlinear analysis of the 10-bar plane frame with the SAP2000 program, the steps in the analysis of the 56-bar truss system were followed. Again, the analysis type was selected as Nonlinear, and the geometric nonlinearity parameter was selected as "P-Delta plus Large Displacements".

In the displacement-controlled Pushover analysis, the height of the frame is 35.5 m, and by default, $35.5 \times 0.04 = 1.42 \text{ m}$ displacement is allowed at the top point. To achieve the real results of the geometrically nonlinear analysis, the top point horizontal displacement $U_1 = 5.218 \text{ cm}$ was taken as the limit value. Displacement-controlled pushover analysis was completed in 10 steps. It has been observed that the nonlinear analysis results obtained with the SAP2000 program are the same as the Stability Functions and Tangent Stiffness Method Algorithm results.

5. Conclusions

In practice, no structural system exhibits linear behavior in terms of both material and geometry. Stability Functions and Tangent Stiffness Method have an important place among the nonlinear solution algorithms.

In this study, iterative computational algorithms based on the Stability Functions and Tangent Stiffness methods were developed to account for the geometrically nonlinear behavior of structures. Using these algorithms, two plane frame structures and one space truss system were analyzed, and the obtained results were compared with benchmark solutions available in the literature. A step-by-step linearization approach was adopted in the proposed methods, and both algorithms produced accurate and consistent numerical results for all benchmark examples.

In the geometrically nonlinear analysis with the SAP2000 program, when the analysis type was selected as Nonlinear and the geometric nonlinearity parameter was selected as “P-Delta plus Large Displacements”, it was observed that the results obtained were the same as the results in the stability functions and tangent stiffness method solutions.

For the benchmark problems considered, the Stability Functions and Tangent Stiffness algorithms successfully captured the geometrically nonlinear behavior associated with large displacements. Comparisons with the benchmark solutions available in the literature and the SAP2000 analyses confirmed the accuracy, consistency, and reliability of the proposed algorithms. Furthermore, the displacement-controlled pushover analysis contributed to the comparative validation of the proposed algorithms up to the reference displacement level.

A displacement-controlled pushover analysis was performed incrementally to evaluate the geometrically nonlinear response of the structure, and the maximum roof displacement corresponding to the reference performance point was adopted for the comparative assessment. The proposed algorithms provide reliable predictions of structural behavior by accounting for large displacements and second-order effects under the assumption of linear-elastic material behavior. At the reference displacement level considered in this study, the structural system maintained its overall stability, and no loss of stability was observed.

As can be seen from the geometrically nonlinear analysis results, there have been major changes in displacements and bar forces. While some bars were compression bars in linear analysis, they turned into tension bars in nonlinear analysis. While some joints were displaced in the positive direction of the global axis in linear analysis, they were displaced in the negative direction as a result of nonlinear analysis. These results clearly show how important geometrically nonlinear behavior is in structural analysis.

Although only geometrically nonlinear analysis was considered in this study, it will be possible to develop an efficient algorithm with one of the programming languages in SAP2000 OAPI [27] by adding optimum design to it in future studies.

Declarations

Conflict of interests

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Author contributions

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Data availability statement

The data presented in this study are available on request from the corresponding author.

Use of generative AI and AI-assisted technologies

The author(s) confirm the author(s) did not use any AI tools in the preparation of this work/research/study.

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