Journal of Structural Engineering & Applied Mechanics (2025) 8(2):152-174
DOI 10.31462/jseam.2025.02152174

RESEARCH ARTICLE

Linear theory of thermoelastic diffusion in triple porosity materials with microtemperatures and microconcentrations

Tarun Kansal*

Markanda National College, Department of Mathematics, Shahabad (M.), India

Article History

Received 19 June 2025
Accepted 6 July 2025

Keywords

Thermoelastic diffusion
Triple porosity
Steady oscillations

Abstract

This paper seeks to formulate the fundamental governing equations for an anisotropic thermoelastic medium distinguished by triple porosity, microtemperatures, and microconcentrations. The model proposed in this study integrates the influences of porosity, microtemperatures, and microconcentrations, crucial factors in providing a precise description of material behavior. Additionally, the objective is to develop the fundamental solution for the system of equations corresponding to steady oscillations and equilibrium conditions.

1. Introduction

In a material with triple porosity, the body displays three distinct levels of pore structures, each representing a different scale of porosity within the material. The first level is termed macro porosity, indicating the largest visible pores within the material that are typically observable to the naked eye or through macroscopic imaging techniques. The second level is referred to as meso porosity, corresponding to an intermediate scale of porosity with pores smaller than macro porosity but larger than micro porosity. The final level is known as micro porosity, representing the smallest scale of pores within the material, characterized by microscopic dimensions that may require high-resolution imaging techniques or advanced microscopic analysis for direct observation.

The existence of these three levels of porosity facilitates a more comprehensive characterization of the material's permeability, transport properties, and mechanical behavior, taking into account variations in pore sizes and their distribution throughout the material. The research of Svanadze [1], Straughan [2], and Kansal [3] has significantly contributed to establishing governing equations in the field of elasticity and thermoelasticity, specifically addressing the concept of triple porosity. Svanadze [4-9] further delved into investigating boundary value problems related to elastic solids and thermoelastic solids with triple porosity. In his book, Straughan [10] discussed various applications of multiple porosities, shedding light on the practical significance of these theories across different disciplines.

Grot [11] expanded the theory of thermodynamics for elastic bodies with microstructure by considering microelements with different temperatures. The Clausius-Duhem inequality was modified to incorporate microtemperatures, and first-order moment of energy equations were introduced into the basic balance laws. These modifications were aimed at determining the microtemperatures of a continuum. Iesan and Quintanilla

* Corresponding author (tarun1_kansal@yahoo.co.in
eISSN 2630-5763

[12] developed a linear theory for elastic materials with an inner structure that includes microtemperatures, in addition to the classical displacement and temperature fields. Their focus was on the existence of solutions for initial boundary value problems, employing semigroup theory. They successfully proved an existence theorem and established the continuous dependence of solutions on the initial data and body loads. Aouadi et al. [13] developed a nonlinear theory of thermoelastic diffusion materials with microtemperatures and microconcentrations, alongside a linear theory for the same materials. The well-posedness of the linear anisotropic problem was proven using semigroup theory, and the asymptotic behavior of solutions was studied. Chiril and Marin [14] derived the field equations and consecutive equations of the linear theory of microstretch thermoelasticity for materials with particles having microelements equipped with microtemperatures and microconcentrations.

The utilization of elementary functions in constructing fundamental solutions enables the development of analytical and numerical methods for solving boundary value problems in elasticity and thermoelasticity. These methods are valuable for providing accurate and efficient solutions across a broad spectrum of engineering and scientific applications. By employing fundamental solutions, researchers gain the capability to explore the behavior of complex systems and analyze the effects of various parameters, boundary conditions, and material properties on the system's response. In the specific context of elasticity and thermoelasticity with triple porosity or microtemperatures, Svanadze [1,15-17] and Kansal [3] have made significant contributions by constructing fundamental solutions using elementary functions. These contributions enhance the understanding and modeling of materials with intricate pore structures or temperature distributions, facilitating advancements in the field and enabling the study of diverse phenomena.

The current paper is structured into several sections, each addressing specific aspects of the study. The constitutive relations and field equations for anisotropic thermoelastic diffusion bodies with triple porosity, microtemperatures, and microconcentrations are derived in Section 2. The system of linearized equations for steady oscillations in the theory of thermoelastic diffusion solids with triple porosity, microtemperatures, and microconcentrations is obtained in Section 3. In Section 4, the fundamental solutions of the basic governing equations are constructed using elementary functions. Some basic properties of the fundamental matrix in the case of steady oscillations are discussed in Section 5. Finally, in Section 6, the fundamental solutions of the basic governing equations in equilibrium conditions are constructed as a particular case. These properties provide information about the behavior of the solutions and can be used to analyze and interpret the results obtained from the fundamental solutions. These sections provide a comprehensive framework for studying and solving boundary value problems in this field and contribute to a deeper understanding of the underlying phenomena.

2. Basic equations

The law of conservation of energy for an arbitrary material volume V bounded by a surface A at time t can be written as

$$\int_V \rho\left[\dot{u}_i\ddot{u}_i + \kappa_1\dot{\nu}_1\ddot{\nu}_1 + \kappa_2\dot{\nu}_2\ddot{\nu}_2 + \kappa_3\dot{\nu}_3\ddot{\nu}_3 + \dot{U}\right] dV = \int_V \rho[\tilde{F}_i\dot{u}_i + \Lambda_i\dot{\nu}_i]\, dV + \int_A [f_i\dot{u}_i + \Omega_{ij}\varpi_j\dot{\nu}_i + q_i\varpi_i]\, dA \quad (1)$$

where ρ is the density, u_i are the components of the displacement vector u, $\tilde{F}_i$ are the components of the external forces per unit mass, U is the internal energy per unit mass, f_i are the components of the surface traction vector **f** occurring on the surface A,$\nu_i$are the volume fraction fields corresponding to macro-, meso- and micro-pores respectively, κ_i are the coefficients of equilibrated inertia, Λ_i are extrinsic equilibrated body forces per unit mass associated with macro-, meso-, and micro-pores, respectively, Ω_{ij} are the components of equilibrated stress vectors corresponding to ν_i measured per unit area of surface A, respectively, q_i are the components of the heat flux vector, ϖ_i are the components of the outward unit normal vector ϖ to the surface A.

The components f_i are related to the stress vectors by the relation

$$f_i = \sigma_{ji}\varpi_j \tag{2}$$

Utilizing Eq. 2 within Eq. 1 and employing the divergence theorem, we obtain

$$\begin{aligned}\int_V \rho\left[\dot{u}_i\ddot{u}_i + \kappa_1\dot{v}_1\ddot{v}_1 + \kappa_2\dot{v}_2\ddot{v}_2 + \kappa_3\dot{v}_3\ddot{v}_3 + \dot{U}\right] dV \\ = \int_V \rho[\tilde{F}_i\dot{u}_i + \Lambda_i\dot{v}_i]\, dV + \int_A [\sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij,j}\dot{v}_i + \Omega_{ij}\dot{v}_{i,j} + q_{i,i}]\, dV\end{aligned} \tag{3}$$

As Eq. 3 holds for every segment of the body, thus, the local expression of the conservation of energy is derived as follows:

$$\rho[\dot{u}_i\ddot{u}_i + \kappa_1\dot{v}_1\ddot{v}_1 + \kappa_2\dot{v}_2\ddot{v}_2 + \kappa_3\dot{v}_3\ddot{v}_3 + \dot{U} = \rho[\tilde{F}_i\dot{u}_i + \Lambda_i\dot{v}_i] + \sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij,j}\dot{v}_i + \Omega_{ij}\dot{v}_{i,j} + q_{i,i} \tag{4}$$

Eq. 4 remains valid upon substituting $\dot{u}_i$ with $\dot{u}_i + \wp_i$, where $\wp_i$ are arbitrary constants, while keeping all other terms unchanged. Therefore, from Eq. 4, we obtain:

$$\begin{aligned}\rho\left[(\dot{u}_i + \wp_i)\ddot{u}_i + \kappa_1\dot{v}_1\ddot{v}_1 + \kappa_2\dot{v}_2\ddot{v}_2 + \kappa_3\dot{v}_3\ddot{v}_3 + \dot{U}\right] \\ = \rho[\tilde{F}_i(\dot{u}_i + \wp_i) + \Lambda_i\dot{v}_i] + \sigma_{ji,j}(\dot{u}_i + \wp_i) + \sigma_{ji}\dot{u}_{i,j} + \Omega_{ij,j}\dot{v}_i + \Omega_{ij}\dot{v}_{i,j} + q_{i,i}.\end{aligned} \tag{5}$$

By subtracting Eq. 4 from Eq. 5, we derive:

$$\wp_i[\sigma_{ji,j} + \rho\tilde{F}_i - \rho \rightleftarrows \ddot{u}_i] = 0 \tag{6}$$

Since the quantities within the square brackets are independent of $\wp_i$ according to Eq. 6, we deduce:

$$\sigma_{ji,j} + \rho\tilde{F}_i = \rho \rightleftarrows \ddot{u}_i \tag{7}$$

With the aid of Eq. 7, Eq. 4 leads to a simplified form of the conservation of energy:

$$\rho \rightleftarrows \dot{U} = \sigma_{ij}\dot{u}_{i,j} + q_{i,i} + \Omega_{ij}\dot{v}_{i,j} - \hbar_i\dot{v}_i \tag{8}$$

where $\hbar_i, i = 1,2,3$ satisfy the relation

$$\begin{aligned}\Omega_{1j,j} + \hbar_1 + \rho\Lambda_1 = \rho\kappa_1\ddot{v}_1, \\ \Omega_{2j,j} + \hbar_2 + \rho\Lambda_2 = \rho\kappa_2\ddot{v}_2, \\ \Omega_{3j,j} + \hbar_3 + \rho\Lambda_3 = \rho\kappa_3\ddot{v}_3\end{aligned} \tag{9}$$

Let ε_i, Ω_i denote the first moments of energy vector and mass diffusion, respectively. The balances for the first moment of energy and mass diffusion are given by:

$$\rho\dot{\varepsilon}_i = q_{ji,j} + q_i - \tilde{\varsigma}_i \tag{10}$$

$$\rho\dot{\Omega}_i = \eta_{ji,j} + \eta_i - \sigma_i \tag{11}$$

where η_i are the components of mass flux vectors, q_{ij}, η_{ij} are, respectively, the first heat flux and mass diffusion moment tensors, $\tilde{\varsigma}_i, \sigma_i$ are, respectively, the microheat flux and micromass flux averages.

The local form of the principle of entropy is typically represented by the following expression:

$$\rho\dot{S} - \left(\frac{q_j}{T} + \frac{q_{ji}T_i}{T}\right)_{,j} + \left(\frac{P\eta_j}{T} + \frac{P\eta_{ji}T_i}{T}\right)_{,j} \geq 0$$

which implies

$$\rho\dot{S}T - q_{j,j} + \frac{q_j T_{,j}}{T} - q_{ji,j}T_i + \frac{q_{ji,j}T_i T_{,j}}{T} - q_{ji,j}T_{i,j} + P_{,j}\eta_j - \frac{P\eta_j T_{,j}}{T} + P\eta_{j,j} + T\left(\frac{P\eta_{ji}T_i}{T}\right)_{,j} \geq 0 \tag{12}$$

where S, P are the entropy and chemical potential per unit mass, respectively, T is the absolute temperature, and T_i is the microtemperature vector.

The local form of the mass concentration law is

$$\eta_{j,j} = \dot{C} \tag{13}$$

where C is the concentration of the diffusion material in the elastic body. For each micro element, the mass conservation law becomes

$$\dot{C} = \left(\eta_j + C_i\eta_{ji}\right)_{,j} = \eta_{j,j} + C_{i,j}\eta_{ji} + C_i\eta_{ji,j} \tag{14}$$

Eq. 8 with the help of Eqs. 10, 11, 14, and the inequality in Eq. 12 becomes

$$\begin{aligned}\rho[T\dot{S} - \dot{U} - T_i\dot{\varepsilon}_i - C_i\dot{\Omega}_i] + \sigma_{ij}\dot{e}_{ij} + \Omega_{ij}\dot{\nu}_{i,j} - \hbar_i\dot{\nu}_i + \frac{1}{T}q_i T_{,i} + \frac{1}{T}T_{,j}q_{ji}T_i - q_{ji}T_{i,j} + P\dot{C} + (q_i - \tilde{\varsigma}_i)T_i \\ - PC_i\eta_{ji,j} - PC_{i,j}\eta_{ji} + P_{,j}\eta_j - \frac{1}{T}P\eta_j T_{,j} + T\left(\frac{P}{T}T_i\eta_{ji}\right)_{,j} + \eta_{ji,j}C_i + (\eta_i - \sigma_i)C_i \geq 0\end{aligned} \tag{15}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are the components of the strain tensor. Introducing the function $\psi = U + T_i\varepsilon_i + C_i\Omega_i - TS$, the inequality in Eq. 15 in the context of linear theory can be expressed as:

$$\begin{aligned}-\rho[\dot{\psi} + S\dot{T} - \dot{T}_i\varepsilon_i - \dot{C}_i\Omega_i] + \sigma_{ij}\dot{e}_{ij} + \Omega_{ij}\dot{\nu}_{i,j} - \hbar_i\dot{\nu}_i + \frac{1}{T}q_i T_{,i} - q_{ji}T_{i,j} + P\dot{C} + (q_i - \tilde{\varsigma}_i)T_i + P_{,j}\eta_j + \eta_{ji,j}C_i \\ + (\eta_i - \sigma_i)C_i \geq 0\end{aligned} \tag{16}$$

The function ψ can be written in terms of independent variables $e_{ij}, \nu_i, \nu_{i,j}, T, T_{,i}, T_i, T_{i,j}, C, C_{,i}, C_i$ and $C_{i,j}$. Therefore, we obtain:

$$\begin{aligned}\dot{\psi} = \frac{\partial\psi}{\partial e_{ij}}\dot{e}_{ij} + \frac{\partial\psi}{\partial\nu_i}\dot{\nu}_i + \frac{\partial\psi}{\partial\nu_{i,j}}\dot{\nu}_{i,j} + \frac{\partial\psi}{\partial T}\dot{T} + \frac{\partial\psi}{\partial T_{,i}}\dot{T}_{,i} + \frac{\partial\psi}{\partial T_i}\dot{T}_i + \frac{\partial\psi}{\partial T_{i,j}}\dot{T}_{i,j} + \frac{\partial\psi}{\partial C}\dot{C} + \frac{\partial\psi}{\partial C_{,i}}\dot{C}_{,i} + \frac{\partial\psi}{\partial C_i}\dot{C}_i \\ + \frac{\partial\psi}{\partial C_{i,j}}\dot{C}_{i,j}\end{aligned} \tag{17}$$

Inequality in Eq. 16 with the help of Eq. 17 becomes

$$\begin{aligned}\left[\sigma_{ij} - \rho\frac{\partial\psi}{\partial e_{ij}}\right]\dot{e}_{ij} + \left[\Omega_{ij} - \rho\frac{\partial\psi}{\partial\nu_{i,j}}\right]\dot{\nu}_{i,j} - \left[\hbar_i + \rho\frac{\partial\psi}{\partial\nu_i}\right]\dot{\nu}_i + \rho\left[\varepsilon_i - \frac{\partial\psi}{\partial T_i}\right]\dot{T}_i + \rho\left[\Omega_i - \frac{\partial\psi}{\partial C_i}\right]\dot{C}_i \\ - \rho\left[S + \frac{\partial\psi}{\partial T}\right]\dot{T} + \left[P - \rho\frac{\partial\psi}{\partial C}\right]\dot{C} - \rho\frac{\partial\psi}{\partial T_{,i}}\dot{T}_{,i} - \rho\frac{\partial\psi}{\partial T_{i,j}}\dot{T}_{i,j} - \rho\frac{\partial\psi}{\partial C_{,i}}\dot{C}_{,i} - \rho\frac{\partial\psi}{\partial C_{i,j}}\dot{C}_{i,j} \\ + \frac{1}{T}q_i T_{,i} - q_{ji}T_{i,j} + (q_i - \tilde{\varsigma}_i)T_i + P_{,j}\eta_j + \eta_{ji,j}C_i + (\eta_i - \sigma_i)C_i \geq 0\end{aligned}$$

Let us introduce the notations $\varphi = \nu - \nu_0, \theta = T - T_0$, where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, T_0 is the reference temperature of the body chosen such that $\left|\frac{\theta}{T_0}\right| << 1$, ν_0 are the volume fraction fields in the reference configuration. In the linear theory, the independent variables are $e_{ij}, \varphi_i, \varphi_{i,j}, \theta, \theta_{,i}, T_i, T_{i,j}, C, C_{,i}, C_i$ and $C_{i,j}$. The above inequality must hold for all rates $\dot{e}_{ij}, \dot{\varphi}_i, \dot{\varphi}_{i,j}, \dot{\theta}, \dot{\theta}_{,i}, \dot{T}_i, \dot{T}_{i,j}, \dot{C}, \dot{C}_{,i}, \dot{C}_i$ and $\dot{C}_{i,j}$. Therefore, the coefficients of the aforementioned variables must equate to zero, indicating:

$$\sigma_{ij}=\rho\frac{\partial\psi}{\partial e_{ij}},\qquad \Omega_{ij}=\rho\frac{\partial\psi}{\partial\varphi_{i,j}},\qquad \hbar_i=-\rho\frac{\partial\psi}{\partial\varphi_i},\qquad \varepsilon_i=\frac{\partial\psi}{\partial T_i},\qquad \Omega_i=\frac{\partial\psi}{\partial C_i},$$
$$S=-\frac{\partial\psi}{\partial\theta},\qquad P=\rho\frac{\partial\psi}{\partial C},\qquad \frac{\partial\psi}{\partial\theta_{,i}}=0,\qquad \frac{\partial\psi}{\partial T_{i,j}}=0,\qquad \frac{\partial\psi}{\partial C_{,i}}=0,\qquad \frac{\partial\psi}{\partial C_{i,j}}=0, \tag{18}$$
$$\frac{1}{T_0}q_i\theta_{,i}-q_{ji}T_{i,j}+(q_i-\tilde{\varsigma}_i)T_i+P_{,j}\eta_j+\eta_{ji,j}C_i+(\eta_i-\sigma_i)C_i\geq 0$$

In the linear theory of materials possessing a centre of symmetry, we can take ψ in the form:

$$\begin{aligned}2\rho\psi={}&c_{ijpl}e_{ij}e_{pl}+2a_{ij}e_{ij}\theta+2b_{ij}e_{ij}C+2c_{ij}e_{ij}\varphi_1+2d_{ij}e_{ij}\varphi_2+2f_{ij}e_{ij}\varphi_3-2l_i\varphi_i\theta-2n_i\varphi_i-\frac{\rho C_e\theta^2}{T_0}\\&-2a\theta C+bC^2+\alpha_i\varphi_i^2+2\alpha_4\varphi_1\varphi_2+2\alpha_5\varphi_2\varphi_3+2\alpha_6\varphi_3\varphi_1+A_{ij}\varphi_{1,i}\varphi_{1,j}\\&+B_{ij}\varphi_{2,i}\varphi_{2,j}+C_{ij}\varphi_{3,i}\varphi_{3,j}+2D_{ij}\varphi_{1,i}\varphi_{2,j}+2E_{ij}\varphi_{2,i}\varphi_{3,j}+2S_{ij}\varphi_{3,i}\varphi_{1,j}-\alpha_{ij}T_iT_j\\&-\beta_{ij}C_iC_j-2\gamma_{ij}T_iC_j-2P_{ij}\varphi_{1,j}T_i-2U_{ij}\varphi_{2,j}T_i-2Q_{ij}\varphi_{3,j}T_i-2M_{ij}\varphi_{1,j}C_i-2T_{ij}\varphi_{2,j}C_i\\&-2\chi_{ij}\varphi_{3,j}C_i\end{aligned}$$

where c_{ijpl} is the tensor of elastic constants, a_{ij}, b_{ij} are, respectively, the tensors of thermal and diffusion expansions. $A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij}, S_{ij}$ and $\alpha_i, i=1,\ldots\ldots,6$ are functions which are typical in porous theories. l_i, n_i are coupling constants. α_{ij}, β_{ij} are, respectively, the tensors of microtemperature and microconcentration effects. $\gamma_{ij}, P_{ij}, U_{ij}, Q_{ij}, M_{ij}, T_{ij}, \chi_{ij}$ are coupling tensors. The consecutive coefficients have the following symmetries:

$$\begin{aligned}&c_{ijpl}=c_{jipl}=c_{plji}, a_{ij}=a_{ji}, b_{ij}=b_{ji}, c_{ij}=c_{ji}, d_{ij}=d_{ji}, f_{ij}=f_{ji}, A_{ij}=A_{ji},\\&B_{ij}=B_{ji}, C_{ij}=C_{ji}, D_{ij}=D_{ji}, E_{ij}=E_{ji}, S_{ij}=S_{ji}, \alpha_{ij}=\alpha_{ji}, \beta_{ij}=\beta_{ji},\\&\gamma_{ij}=\gamma_{ji}, P_{ij}=P_{ji}, U_{ij}=U_{ji}, Q_{ij}=Q_{ji}, M_{ij}=M_{ji}, T_{ij}=T_{ji}, \chi_{ij}=\chi_{ji}\end{aligned}$$

Using the above equation in the system of Eq. 18, the following consecutive equations are obtained:

$$\begin{aligned}\sigma_{ij}&=c_{ijpl}e_{pl}+c_{ij}\varphi_1+d_{ij}\varphi_2+f_{ij}\varphi_3+a_{ij}\theta+b_{ij}C,\\\Omega_{1j}&=A_{ij}\varphi_{1,i}+D_{ij}\varphi_{2,i}+S_{ij}\varphi_{3,i}-P_{ij}T_i-M_{ij}C_i,\\\Omega_{2j}&=D_{ij}\varphi_{1,i}+B_{ij}\varphi_{2,i}+E_{ij}\varphi_{3,i}-U_{ij}T_i-T_{ij}C_i,\\\Omega_{3j}&=S_{ij}\varphi_{1,i}+E_{ij}\varphi_{2,i}+C_{ij}\varphi_{3,i}-Q_{ij}T_i-\chi_{ij}C_i,\\\hbar_1&=-c_{ij}e_{ij}-\alpha_1\varphi_1-\alpha_4\varphi_2-\alpha_6\varphi_3+l_1\theta+n_1C,\\\hbar_2&=-d_{ij}e_{ij}-\alpha_4\varphi_1-\alpha_2\varphi_2-\alpha_5\varphi_3+l_2\theta+n_2C,\\\hbar_3&=-f_{ij}e_{ij}-\alpha_6\varphi_1-\alpha_5\varphi_2-\alpha_3\varphi_3+l_3\theta+n_3C,\\\rho\varepsilon_i&=-P_{ij}\varphi_{1,j}-U_{ij}\varphi_{2,j}-Q_{ij}\varphi_{3,j}-\alpha_{ij}T_j-\gamma_{ij}C_j,\\\rho\Omega_i&=-M_{ij}\varphi_{1,j}-T_{ij}\varphi_{2,j}-\chi_{ij}\varphi_{3,j}-\gamma_{ji}T_j-\beta_{ij}C_j\end{aligned} \tag{19}$$

$$\rho S=-a_{ij}e_{ij}+l_i\varphi_i+\frac{\rho C_e\theta}{T_0}+aC, P=b_{ij}e_{ij}-n_i\varphi_i-a\theta+bC,$$

The linear expressions for $q_i, q_{ij}, \tilde{\varsigma}_i, \eta_i, \eta_{ij}, \sigma_i$ are

$$\begin{aligned}&q_i=k_{ij}\theta_{,j}+\kappa_{ij}T_j, q_{ij}=-m_{ijpg}T_{g,p}, \tilde{\varsigma}_i=(k_{ij}-K_{ij})\theta_{,j}+(\kappa_{ij}-L_{ij})T_j,\\&\eta_i=h_{ij}P_{,j}+\tilde{h}_{ij}C_j, \eta_{ij}=-n_{ijpg}C_{g,p}, \sigma_i=(h_{ij}-H_{ij})P_{,j}+(\tilde{h}_{ij}-\tilde{H}_{ij})C_j\end{aligned} \tag{20}$$

where the constitutive coefficients $k_{ij}, \kappa_{ij}, K_{ij}, L_{ij}, h_{ij}, \tilde{h}_{ij}, H_{ij}, \tilde{H}_{ij}, m_{ijpg}$ and n_{ijpg} satisfy the inequality

$$\frac{1}{T_0}k_{ij}\theta_{,i}\theta_{,j}+\left(K_{ij}+\frac{1}{T_0}\kappa_{ji}\right)\theta_{,j}T_i+L_{ij}T_iT_j+m_{jipg}T_{g,p}T_{i,j}+h_{ij}P_{,i}P_{,j}+(H_{ij}+\tilde{h}_{ji})P_{,j}C_i+\widetilde{H}_{ij}C_iC_j - n_{jipg}C_{g,pj}C_i\geq 0$$

and the following symmetries

$$k_{ij}=k_{ji},\quad \kappa_{ij}=\kappa_{ji},\quad K_{ij}=K_{ji},\quad L_{ij}=L_{ji},\quad h_{ij}=h_{ji},\quad \tilde{h}_{ij}=\tilde{h}_{ji},\quad H_{ij}=H_{ji},\quad \widetilde{H}_{ij}=\widetilde{H}_{ji},\quad m_{ijpg}=m_{jipg},\quad n_{ijpg}=n_{jipg}$$

The linearized form of inequality in Eq. 12 is

$$\rho T_0\dot{S}=q_{i,i} \tag{21}$$

Considering Eqs. 19 and 20, Eqs. 7, 9-11, 13, and 21 can be reformulated as:

$$\begin{gathered}
c_{ijpl}e_{pl,j}+c_{ij}\varphi_{1,j}+d_{ij}\varphi_{2,j}+f_{ij}\varphi_{3,j}+a_{ij}\theta_{,j}+b_{ij}C_{,j}+\rho\tilde{F}_i=\rho\ddot{u}_i,\\
m_{jipg}T_{g,pj}-\alpha_{ij}\dot{T}_j-\gamma_{ij}\dot{C}_j-P_{ij}\dot{\varphi}_{1,j}-U_{ij}\dot{\varphi}_{2,j}-Q_{ij}\dot{\varphi}_{3,j}=K_{ij}\theta_{,j}+L_{ij}T_j,\\
n_{jipg}C_{g,pj}-\gamma_{ji}\dot{T}_j-\beta_{ij}\dot{C}_j-M_{ij}\dot{\varphi}_{1,j}-T_{ij}\dot{\varphi}_{2,j}-\chi_{ij}\dot{\varphi}_{3,j}=H_{ij}P_{,j}+\widetilde{H}_{ij}C_j,\\
-c_{ij}e_{ij}-P_{ij}T_{i,j}-M_{ij}C_{i,j}+A_{ij}\varphi_{1,ij}+D_{ij}\varphi_{2,ij}+S_{ij}\varphi_{3,ij}-\alpha_1\varphi_1-\alpha_4\varphi_2\\
-\alpha_6\varphi_3+l_1\theta+n_1C+\rho\Lambda_1=\rho\kappa_1\ddot{\varphi}_1,\\
-d_{ij}e_{ij}-U_{ij}T_{i,j}-T_{ij}C_{i,j}+D_{ij}\varphi_{1,ij}+B_{ij}\varphi_{2,ij}+E_{ij}\varphi_{3,ij}-\alpha_4\varphi_1-\alpha_2\varphi_2\\
-\alpha_5\varphi_3+l_2\theta+n_2C+\rho\Lambda_2=\rho\kappa_2\ddot{\varphi}_2,\\
-f_{ij}e_{ij}-Q_{ij}T_{i,j}-\chi_{ij}C_{i,j}+S_{ij}\varphi_{1,ij}+E_{ij}\varphi_{2,ij}+C_{ij}\varphi_{3,ij}-\alpha_6\varphi_1-\alpha_5\varphi_2\\
-\alpha_3\varphi_3+l_3\theta+n_3C+\rho\Lambda_3=\rho\kappa_3\ddot{\varphi}_3,\\
T_0[-a_{ij}\dot{e}_{ij}+l_i\dot{\varphi}_i+a\dot{C}]+\rho C_e\dot{\theta}=k_{ij}\theta_{,ij}+\kappa_{ij}T_{j,i},\\
h_{ij}[b_{pl}e_{pl}-n_g\varphi_g-a\theta+bC]_{,ij}+\tilde{h}_{ij}C_{j,i}=\dot{C}
\end{gathered}\tag{22}$$

For an isotropic and homogeneous material, the subsequent equations are simplified to:

$$\begin{gathered}
\sigma_{ij}=\lambda e_{pp}\delta_{ij}+2\mu e_{ij}+[\tilde{\lambda}_p\varphi_p-\beta_1\theta-\beta_2C]\delta_{ij},\\
\Omega_{1j}=A_1\varphi_{1,j}+A_4\varphi_{2,j}+A_6\varphi_{3,j}-B_1T_j-B_4C_j,\\
\Omega_{2j}=A_4\varphi_{1,j}+A_2\varphi_{2,j}+A_5\varphi_{3,j}-B_2T_j-B_5C_j,\\
\Omega_{3j}=A_6\varphi_{1,j}+A_5\varphi_{2,j}+A_3\varphi_{3,j}-B_3T_j-B_6C_j,\\
\hbar_1=-\tilde{\lambda}_1e_{pp}-\alpha_1\varphi_1-\alpha_4\varphi_2-\alpha_6\varphi_3+l_1\theta+n_1C,\\
\hbar_2=-\tilde{\lambda}_2e_{pp}-\alpha_4\varphi_1-\alpha_2\varphi_2-\alpha_5\varphi_3+l_2\theta+n_2C,\\
\hbar_3=-\tilde{\lambda}_3e_{pp}-\alpha_6\varphi_1-\alpha_5\varphi_2-\alpha_3\varphi_3+l_3\theta+n_3C,\\
\rho\varepsilon_i=-B_1\varphi_{1,i}-B_2\varphi_{2,i}-B_3\varphi_{3,i}-D_1T_i-D_3C_i,\\
\rho\Omega_i=-B_4\varphi_{1,i}-B_5\varphi_{2,i}-B_6\varphi_{3,i}-D_3T_i-D_2C_i,\\
\rho S=\beta_1e_{pp}+l_i\varphi_i+\frac{\rho C_e\theta}{T_0}+aC,P=-\beta_2e_{pp}-n_i\varphi_i-a\theta+bC,\\
q_i=k\theta_{,i}+k_1T_i,q_{ij}=-k_4T_{p,p}\delta_{ij}-k_5T_{i,j}-k_6T_{j,i},\tilde{\varsigma}_i=(k-k_3)\theta_{,i}+(k_1-k_2)T_i,\\
\eta_i=hP_{,i}+h_1C_i,\eta_{ij}=-h_4C_{p,p}\delta_{ij}-h_5C_{i,j}-h_6C_{j,i},\sigma_i=(h-h_3)P_{,i}+(h_1-h_2)C_i
\end{gathered}\tag{23}$$

where

$$\begin{gathered}
c_{ijpl}=\lambda\delta_{ij}\delta_{pl}+\mu\delta_{ip}\delta_{jl}+\mu\delta_{il}\delta_{jp},a_{ij}=-\beta_1\delta_{ij},b_{ij}=-\beta_2\delta_{ij},c_{ij}=\tilde{\lambda}_1\delta_{ij},d_{ij}=\tilde{\lambda}_2\delta_{ij},f_{ij}=\tilde{\lambda}_3\delta_{ij},\\
A_{ij}=A_1\delta_{ij},B_{ij}=A_2\delta_{ij},C_{ij}=A_3\delta_{ij},D_{ij}=A_4\delta_{ij},E_{ij}=A_5\delta_{ij},S_{ij}=A_6\delta_{ij},P_{ij}=B_1\delta_{ij},\\
U_{ij}=B_2\delta_{ij},Q_{ij}=B_3\delta_{ij},M_{ij}=B_4\delta_{ij},T_{ij}=B_5\delta_{ij},\chi_{ij}=B_6\delta_{ij},\alpha_{ij}=D_1\delta_{ij},\beta_{ij}=D_2\delta_{ij},\gamma_{ij}=D_3\delta_{ij},\\
k_{ij}=k\delta_{ij},\kappa_{ij}=k_1\delta_{ij},L_{ij}=k_2\delta_{ij},K_{ij}=k_3\delta_{ij},h_{ij}=h\delta_{ij},\tilde{h}_{ij}=h_1\delta_{ij},\widetilde{H}_{ij}=h_2\delta_{ij},H_{ij}=h_3\delta_{ij},\\
m_{ijpl}=k_4\delta_{ij}\delta_{pl}+k_6\delta_{ip}\delta_{jl}+k_5\delta_{il}\delta_{ip},n_{ijpl}=h_4\delta_{ij}\delta_{pl}+h_6\delta_{ip}\delta_{jl}+h_5\delta_{il}\delta_{ip}
\end{gathered}$$

where δ_{ij} is Kronecker's delta and $\lambda, \mu, \beta_1, \beta_2, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, A_1, A_6, B_1, B_6, D_1, D_2, D_3, k, h, k_1, k_6$ and h_1, h_6 are material constants. Therefore, utilizing Eq. 22 in conjunction with the support of Eq. 23, we establish the governing equations for homogeneous isotropic thermoelastic diffusion. These equations incorporate considerations for microtemperatures, microconcentrations, and triple porosity, under the assumption of the absence of heat and mass diffusive sources:

$$
\begin{gathered}
\mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u) + \bar{\lambda}_i\nabla\varphi_i - \beta_1\nabla\theta - \beta_2\nabla C = \rho\ddot{u},\\
(k_6\Delta - k_2)v + (k_4+k_5)\nabla(\nabla\cdot v) - k_3\nabla\theta = D_1\dot{v} + D_3\dot{w} + B_i\nabla\dot{\varphi}_i,\\
(h_6\Delta - h_2)w + (h_4+h_5)\nabla(\nabla\cdot w) - h_3\nabla P = D_3\dot{v} + D_2\dot{w} + B_{i+3}\nabla\dot{\varphi}_i,\\
-\bar{\lambda}_1(\nabla\cdot u) - B_1(\nabla\cdot v) - B_4(\nabla\cdot w) + (A_1\Delta-\alpha_1)\varphi_1 + (A_4\Delta-\alpha_4)\varphi_2 + (A_6\Delta-\alpha_6)\varphi_3\\
+l_1\theta + n_1C = \rho\kappa_1\ddot{\varphi}_1,\\
-\bar{\lambda}_2(\nabla\cdot u) - B_2(\nabla\cdot v) - B_5(\nabla\cdot w) + (A_4\Delta-\alpha_4)\varphi_1 + (A_2\Delta-\alpha_2)\varphi_2 + (A_5\Delta-\alpha_5)\varphi_3\\
+l_2\theta + n_2C = \rho\kappa_2\ddot{\varphi}_2,\\
-\bar{\lambda}_3(\nabla\cdot u) - B_3(\nabla\cdot v) - B_6(\nabla\cdot w) + (A_6\Delta-\alpha_6)\varphi_1 + (A_5\Delta-\alpha_5)\varphi_2 + (A_3\Delta-\alpha_3)\varphi_3\\
+l_3\theta + n_3C = \rho\kappa_3\ddot{\varphi}_3,\\
T_0[\beta_1(\nabla\cdot\dot{u}) + l_i\dot{\varphi}_i + a\dot{C}] + \rho C_e\dot{\theta} = k\Delta\theta + k_1(\nabla\cdot v),\\
h\Delta[-\beta_2(\nabla\cdot u) - n_i\varphi_i - a\theta + bC] + h_1(\nabla\cdot w) = \dot{C}
\end{gathered}
\tag{24}
$$

where $v = (T_1, T_2, T_3)$ and $w = (C_1, C_2, C_3)$ are, respectively, microtemperature and microconcentration vectors, Δ and ∇ are, respectively, Laplacian and Del operators.

In the following sections, the chemical potential has been adopted as a state variable rather than concentration. From the 11th equation of Eq. 23, we get

$$C = \frac{1}{b}\left[P + \beta_2 e_{pp} + n_i\varphi_i + a\theta\right]$$

Hence, the system of Eq. 24, with the assistance of the above equation, is transformed into

$$
\begin{gathered}
\mu\Delta u + (\lambda'+\mu)\nabla(\nabla\cdot u) + \tilde{\lambda}_i\nabla\varphi_i - \vartheta_1\nabla\theta - \vartheta_2\nabla P = \rho\ddot{u},\\
(k_6\Delta - k_2)v + (k_4+k_5)\nabla(\nabla\cdot v) - k_3\nabla\theta = D_1\dot{v} + D_3\dot{w} + B_i\nabla\dot{\varphi}_i,\\
(h_6\Delta - h_2)w + (h_4+h_5)\nabla(\nabla\cdot w) - h_3\nabla P = D_3\dot{v} + D_2\dot{w} + B_{i+3}\nabla\dot{\varphi}_i,\\
-\tilde{\lambda}_1(\nabla\cdot u) - B_1(\nabla\cdot v) - B_4(\nabla\cdot w) + (A_1\Delta-\zeta_1)\varphi_1 + (A_4\Delta-\zeta_4)\varphi_2 + (A_6\Delta-\zeta_6)\varphi_3\\
+\xi_1\theta + v_1P = \rho\kappa_1\ddot{\varphi}_1,\\
-\tilde{\lambda}_2(\nabla\cdot u) - B_2(\nabla\cdot v) - B_5(\nabla\cdot w) + (A_4\Delta-\zeta_4)\varphi_1 + (A_2\Delta-\zeta_2)\varphi_2 + (A_5\Delta-\zeta_5)\varphi_3\\
+\xi_2\theta + v_2P = \rho\kappa_2\ddot{\varphi}_2,\\
-\tilde{\lambda}_3(\nabla\cdot u) - B_3(\nabla\cdot v) - B_6(\nabla\cdot w) + (A_6\Delta-\zeta_6)\varphi_1 + (A_5\Delta-\zeta_5)\varphi_2 + (A_3\Delta-\zeta_3)\varphi_3\\
+\xi_3\theta + v_3P = \rho\kappa_3\ddot{\varphi}_3,\\
-T_0[\vartheta_1(\nabla\cdot\dot{u}) + \xi_i\dot{\varphi}_i + \varsigma\dot{P}] - \eta T_0\dot{\theta} + k_1(\nabla\cdot v) + k\Delta\theta = 0,\\
-\left[\vartheta_2(\nabla\cdot\dot{u}) + v_i\dot{\varphi}_i + \varsigma\dot{\theta} + \varpi\dot{P}\right] + h_1(\nabla\cdot w) + h\Delta P = 0
\end{gathered}
\tag{25}
$$

where

$$
\begin{gathered}
\varpi = b^{-1}, \quad \vartheta_2 = \varpi\beta_2, \quad \vartheta_1 = \beta_1 + a\vartheta_2, \quad \tilde{\lambda}_i = \bar{\lambda}_i - n_i\vartheta_2, \quad \lambda' = \lambda - \vartheta_2\beta_2,\\
\varsigma = a\varpi, v_i = n_i\varpi, \quad \zeta_i = \alpha_i - n_i v_i, \quad \zeta_4 = \alpha_4 - n_1 v_2, \quad \zeta_5 = \alpha_5 - n_2 v_3,\\
\zeta_6 = \alpha_6 - n_3 v_1, \quad \xi_i = l_i + n_i\varsigma, \quad \eta = \frac{\rho C_e}{T_0} + a\varsigma i = 1,2,3
\end{gathered}
$$

3. Steady oscillations

The displacement vector, microtemperature, microconcentration, volume fraction fields, temperature change, and chemical potential functions are presumed as

$$[u(x,t), v(x,t), w(x,t), \varphi(x,t), \theta(x,t), P(x,t)] = \mathrm{Re}[(u^*, v^*, w^*, \varphi^*, \theta^*, P^*)e^{-\iota\omega t}] \tag{26}$$

where, ω is the oscillation frequency. By employing Eq. 26 in the system of Eq. 25 and omitting asterisks (*) for simplicity, we derive the system of equations for steady oscillations as

$$\begin{gathered}
[\mu\Delta + \rho\omega^2]u + (\lambda' + \mu)\nabla(\nabla\cdot u) + \tilde{\lambda}_i\nabla\varphi_i - \vartheta_1\nabla\theta - \vartheta_2\nabla P = 0,\\
[k_6\Delta - k_2 + \iota\omega D_1]v + (k_4 + k_5)\nabla(\nabla\cdot v) + \iota\omega D_3 w + \iota\omega B_i\nabla\varphi_i - k_3\nabla\theta = 0,\\
\iota\omega D_3 v + [h_6\Delta - h_2 + \iota\omega D_2]w + (h_4 + h_5)\nabla(\nabla\cdot w) + \iota\omega B_{i+3}\nabla\varphi_i - h_3\nabla P = 0,\\
-\tilde{\lambda}_1(\nabla\cdot u) - B_1(\nabla\cdot v) - B_4(\nabla\cdot w) + (A_1\Delta - \gamma_1)\varphi_1 + (A_4\Delta - \zeta_4)\varphi_2 + (A_6\Delta - \zeta_6)\varphi_3\\
+\xi_1\theta + v_1 P = 0,\\
-\tilde{\lambda}_2(\nabla\cdot u) - B_2(\nabla\cdot v) - B_5(\nabla\cdot w) + (A_4\Delta - \zeta_4)\varphi_1 + (A_2\Delta - \gamma_2)\varphi_2 + (A_5\Delta - \zeta_5)\varphi_3\\
+\xi_2\theta + v_2 P = 0,\\
-\tilde{\lambda}_3(\nabla\cdot u) - B_3(\nabla\cdot v) - B_6(\nabla\cdot w) + (A_6\Delta - \zeta_6)\varphi_1 + (A_5\Delta - \zeta_5)\varphi_2 + (A_3\Delta - \gamma_3)\varphi_3\\
+\xi_3\theta + v_3 P = 0,\\
\iota\omega T_0[\vartheta_1(\nabla\cdot u) + \xi_i\varphi_i + \varsigma P] + k_1(\nabla\cdot v) + [k\Delta + \iota\omega\eta T_0]\theta = 0,\\
\iota\omega[\vartheta_2(\nabla\cdot u) + v_i\varphi_i + \varsigma\theta] + h_1(\nabla\cdot w) + [h\Delta + \iota\omega\varpi]P = 0
\end{gathered} \tag{27}$$

where, $\gamma_i = \zeta_i - \rho\kappa_i\omega^2, i = 1,2,3$. We introduce the second-order matrix differential operators with constant coefficients

(i) $$\mathbf{F}(\mathbf{D}_x) = \left(\mathbf{F}_{gl}(\mathbf{D}_x)\right)_{14\times14}$$

where $F_{gl}(D_x), g, l = 1, \ldots\ldots, 14$ are given in Appendix A.

(ii) $$\tilde{\mathbf{F}}(\mathbf{D}_x) = \left(\tilde{\mathbf{F}}_{gl}(\mathbf{D}_x)\right)_{14\times14}$$

where $\tilde{F}_{gl}(D_x), g, l = 1, \ldots\ldots, 14$ are given in Appendix A. The system of Eq. 27 can be represented as

$$\mathbf{F}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \boldsymbol{\varphi}, \boldsymbol{\theta}, \mathbf{P})$ is a fourteen-component vector function for E^3. The matrix $\tilde{F}(D_x)$ is called the principal part of the operator $F(D_x)$.

DEFINITION 1: The operator $F(D_x)$ is said to be elliptic if $|\tilde{F}(m)| \neq 0$, where $m = (m_1, m_2, m_3)$. Since

$$|\tilde{F}(m)| = \mu^2\tilde{\lambda}kk_6k_7hh_6h_7\vartheta|m|^{30}, \tilde{\lambda} = \lambda' + 2\mu, k_7 = k_4 + k_5 + k_6, h_7 = h_4 + h_5 + h_6$$

where $\vartheta = \begin{vmatrix} A_1 A_4 A_6 \\ A_4 A_2 A_5 \\ A_6 A_5 A_3 \end{vmatrix}$, therefore operator $F(D_x)$ is an elliptic differential operator if

$$\mu\tilde{\lambda}kk_6k_7hh_6h_7\vartheta \neq 0 \tag{28}$$

DEFINITION 2: The fundamental solution of the system of Eq. 27 (the fundamental matrix of operator F) is the matrix $G(x) = \left(G_{gl}(x)\right)_{14\times14}$ satisfying condition

$$F(D_x)G(x) = \delta(x)I(x) \tag{29}$$

where $\delta(x)$ is the Dirac delta, $I(x) = \left(\delta_{gl}\right)_{14\times14}$ is the unit matrix and $x \in E^3$. Now, we construct $G(x)$ in terms of elementary functions.

4. Construction of $G(x)$ in terms of elementary functions

Let us consider the system of non-homogeneous equations

$$(\mu\Delta + \rho\omega^2)u + (\lambda' + \mu)\nabla(\nabla\cdot u) - \tilde{\lambda}_i\nabla\varphi_i + \iota\omega T_0\vartheta_1\nabla\theta + \iota\omega\vartheta_2\nabla P = H \tag{30}$$

$$(k_6\Delta + k_8)v + (k_4 + k_5)\nabla(\nabla\cdot v) + \iota\omega D_3 w - B_i\nabla\varphi_i + k_1\nabla\theta = V \tag{31}$$

$$\iota\omega D_3 v + (h_6\Delta + h_8)w + (h_4 + h_5)\nabla(\nabla\cdot w) - B_{i+3}\nabla\varphi_i + h_1\nabla P = W \tag{32}$$

$$\begin{gathered}\tilde{\lambda}_1(\nabla\cdot u) + \iota\omega B_1(\nabla\cdot v) + \iota\omega B_4(\nabla\cdot w) + (A_1\Delta - \gamma_1)\varphi_1 + (A_4\Delta - \zeta_4)\varphi_2 + (A_6\Delta - \zeta_6)\varphi_3 \\ +\iota\omega\xi_1 T_0\theta + \iota\omega v_1 P = X_1\end{gathered} \tag{33}$$

$$\begin{gathered}\tilde{\lambda}_2(\nabla\cdot u) + \iota\omega B_2(\nabla\cdot v) + \iota\omega B_5(\nabla\cdot w) + (A_4\Delta - \zeta_4)\varphi_1 + (A_2\Delta - \gamma_2)\varphi_2 + (A_5\Delta - \zeta_5)\varphi_3 \\ +\iota\omega\xi_2 T_0\theta + \iota\omega v_2 P = X_2\end{gathered} \tag{34}$$

$$\begin{gathered}\tilde{\lambda}_3(\nabla\cdot u) + \iota\omega B_3(\nabla\cdot v) + \iota\omega B_6(\nabla\cdot w) + (A_6\Delta - \zeta_6)\varphi_1 + (A_5\Delta - \zeta_5)\varphi_2 + (A_3\Delta - \gamma_3)\varphi_3 \\ +\iota\omega\xi_3 T_0\theta + \iota\omega v_3 P = X_3\end{gathered} \tag{35}$$

$$-\vartheta_1(\nabla\cdot u) - k_3(\nabla\cdot v) + \xi_i\varphi_i + [k\Delta + \iota\omega T_0\eta]\theta + \iota\omega\varsigma P = Y \tag{36}$$

$$-\vartheta_2(\nabla\cdot u) - h_3(\nabla\cdot w) + v_i\varphi_i + \iota\omega T_0\varsigma\theta + [h\Delta + \iota\omega\varpi]P = Z \tag{37}$$

where **H**, **V**, **W** are three-component vector functions on E^3; X_i , Y and Z are scalar functions on E^3. The system of Eqs. 30–37 may be written in the form

$$F^{tr}(D_x)U(x) = Q(x) \tag{38}$$

where F^{tr} is the transpose of matrix $F, Q = (H, V, W, X_i, Y, Z)$, $x \in E^3$. Applying operator (∇) to the Eqs. 30-32, we obtain

$$(\tilde{\lambda}\Delta + \rho\omega^2)\nabla\cdot u - \tilde{\lambda}_i\Delta\varphi_i + \iota\omega T_0\vartheta_1\Delta\theta + \iota\omega\vartheta_2\Delta P = \nabla\cdot H \tag{39}$$

$$(k_7\Delta + k_8)\nabla\cdot v + \iota\omega D_3\nabla\cdot w - B_i\Delta\varphi_i + k_1\Delta\theta = \nabla\cdot V \tag{40}$$

$$\iota\omega D_3\nabla\cdot v + (h_7\Delta + h_8)\nabla\cdot w - B_{i+3}\Delta\varphi_i + h_1\Delta P = \nabla\cdot W \tag{41}$$

Eqs. 33-37 and 39-41 may be expressed in the form

$$N(\Delta) \rightleftarrows\rightleftarrows S = \tilde{Q} \tag{42}$$

where $S, \tilde{Q}, N(\Delta)$ are defined in Appendix B. Eq. 42 may also be written in determinant form as

$$\Gamma_1(\Delta)S = \Psi \tag{43}$$

where $\Psi = (\Psi_1, \ldots\ldots\ldots, \Psi_8)$, $\Psi_p = \frac{1}{\tilde{A}}\sum_{i=1}^{8} N^*_{ip}w_i$, $\Gamma_1(\Delta) = \frac{|N(\Delta)|}{\tilde{A}}$, $\tilde{A} = \tilde{\lambda}kk_7hh_7\vartheta p = 1, \ldots\ldots, 8$ and N^*_{ip} is the cofactor of the element$N_{ip}$of the matrix N. On expanding $\Gamma_1(\Delta)$, We see that

$$\Gamma_1(\Delta) = \prod_{i=1}^{8}(\Delta + \lambda_i^2)$$

where $\lambda_i^2, i = 1, \ldots.., 8$ are the roots of the equation $\Gamma_1(-m) = 0$, for m. Applying operator $\Gamma_1(\Delta)$ to Eq. 31 with the assistance of Eq. 43, we obtain

$$\Gamma_1(\Delta)(\Delta + \lambda_9^2)u = \Psi' \tag{44}$$

where $\lambda_9^2, \Gamma_1(\Delta)$ and Ψ' are given in Appendix C. Multiplying Eqs. 31 and 32 by $h_6\Delta + h_8$ and $\iota\omega D_3$ respectively, we obtain

$$\begin{gathered}(h_6\Delta + h_8)[(k_6\Delta + k_8)v + (k_4 + k_5)\nabla(\nabla\cdot v)] + (h_6\Delta + h_8)\iota\omega D_3 w = \\ (h_6\Delta + h_8)[V + B_i\nabla\varphi_i - k_1\nabla\theta]\end{gathered} \tag{45}$$

$$(\iota\omega D_3)^2 v + \iota\omega D_3[(h_6\Delta + h_8)w + (h_4 + h_5)\nabla(\nabla\cdot w)] = \iota\omega D_3[W + B_{i+3}\nabla\varphi_i - h_1\nabla P] \tag{46}$$

Employing Eq. 46 in Eq. 45, we get

$$\begin{gathered}(h_6\Delta + h_8)[(k_6\Delta + k_8)v + (k_4 + k_5)\nabla(\nabla\cdot v)] - (\iota\omega D_3)^2 v = \\ (h_6\Delta + h_8)[V + B_i\nabla\varphi_i - k_1\nabla\theta] + \iota\omega D_3(h_4 + h_5)\nabla(\nabla\cdot w) - \iota\omega D_3[W + B_{i+3}\nabla\varphi_i - h_1\nabla P]\end{gathered}$$

Applying $\Gamma_1(\Delta)$ to the above equation with the assistance of Eq. 43, we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)v = \Psi'' \tag{47}$$

where $\Gamma_2(\Delta)$ and Ψ'' are mentioned in Appendix C. It can be seen that $\Gamma_2(\Delta) = (\Delta + \lambda_{10}^2)(\Delta + \lambda_{11}^2)$, where $\lambda_i^2, i = 10,11$ are the roots of the equation $\Gamma_2(-m) = 0$, with respect to m. Multiplying Eqs. 31 and 32 by $\iota\omega D_3$ and $k_6\Delta + k_8$ respectively, we obtain

$$\iota\omega D_3[(k_6\Delta + k_8)v + (k_4 + k_5)\nabla(\nabla\cdot v)] + (\iota\omega D_3)^2 w = \iota\omega D_3[V + B_i\nabla\varphi_i - k_1\nabla\theta] \tag{48}$$

$$\begin{gathered}\iota\omega D_3(k_6\Delta + k_8)v + (k_6\Delta + k_8)[(h_6\Delta + h_8)w + (h_4 + h_5)\nabla(\nabla\cdot w)] = \\ (k_6\Delta + k_8)[W + B_{i+3}\nabla\varphi_i - h_1\nabla P]\end{gathered} \tag{49}$$

Using Eq. 48 in Eq. 49, we get

$$\begin{gathered}(k_6\Delta + k_8)[(h_6\Delta + h_8)w + (h_4 + h_5)\nabla(\nabla\cdot w)] - (\iota\omega D_3)^2 w = \\ (k_6\Delta + k_8)[W + B_{i+3}\nabla\varphi_i - h_1\nabla P] + \iota\omega D_3(k_4 + k_5)\nabla(\nabla\cdot v) - \iota\omega D_3[V + B_i\nabla\varphi_i - k_1\nabla\theta]\end{gathered}$$

Applying $\Gamma_1(\Delta)$ to the above equation with the help of Eq. 43, we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)w = \Psi''' \tag{50}$$

where Ψ''' is defined in Appendix C. From Eqs. 43, 44, 47 and 50, we have

$$\Theta(\Delta)U(x) = \widehat{\Psi}(x) \tag{51}$$

where $\widehat{\Psi}(x)$ and $\Theta(\Delta)$ are mentioned in Appendix C. The expressions for $\Psi', \Psi'', \Psi''', \Psi_p, p = 4,\ldots,8$ can be rewritten in the form

$$\begin{aligned}\Psi' &= \frac{1}{\mu}\Gamma_1(\Delta)JH + w_{11}(\Delta)\nabla(\nabla\cdot H) + w_{21}(\Delta)\nabla(\nabla\cdot V) + w_{31}(\Delta)\nabla(\nabla\cdot W) + \sum_{i=4}^{8} w_{i1}(\Delta)\ \nabla w_i,\\ \Psi'' &= w_{12}(\Delta)\nabla(\nabla\cdot H) + \frac{1}{\overline{\overline{B}}}(h_6\Delta + h_8)\Gamma_1(\Delta)JV + w_{22}(\Delta)\nabla(\nabla\cdot V)\\ &\quad - \frac{1}{\overline{\overline{B}}}\iota\omega D_3\Gamma_1(\Delta)JW + w_{32}(\Delta)\nabla(\nabla\cdot W) + \sum_{i=4}^{8} w_{i2}(\Delta)\ \nabla w_i,\\ \Psi''' &= w_{13}(\Delta)\nabla(\nabla\cdot H) - \frac{1}{\overline{\overline{B}}}\iota\omega D_3\Gamma_1(\Delta)JV + w_{23}(\Delta)\nabla(\nabla\cdot V)\\ &\quad + \frac{1}{\overline{\overline{B}}}(k_6\Delta + k_8)\Gamma_1(\Delta)JW + w_{33}(\Delta)\nabla(\nabla\cdot W) + \sum_{i=4}^{8} w_{i3}(\Delta)\ \nabla w_i,\\ \Psi_p &= w_{1p}(\Delta)\nabla\cdot H + w_{2p}(\Delta)\nabla\cdot V + w_{3p}(\Delta)\nabla\cdot W + \sum_{i=4}^{8} w_{ip}(\Delta)\, w_i,\end{aligned} \tag{52}$$

where $J = \left(\delta_{pq}\right)_{3\times3}$ is the unit matrix and $w_{ij}(\Delta), i, j = 1,\ldots,8$ is given in Appendix D. From Eq. 52, we have

$$\widehat{\Psi}(x) = R^{tr}(D_x)Q(x) \tag{53}$$

where $R(D_x) = \left(R_{gq}(D_x)\right)_{14\times 14}$ is given in Appendix D. From Eqs. 38, 51, and 53, we obtain

$$F(D_x)R(D_x) = \Theta(\Delta) \tag{54}$$

We assume that $\lambda_p^2 \neq \lambda_q^2 \neq 0,\ p, q = 1, \ldots, 11$.

Let

$$Y(x) = \left(Y_{ij}(x)\right)_{14\times 14}, \qquad Y_{pp}(x) = \sum_{g=1}^{9} r_{1g}\, \varsigma_g(x),$$
$$Y_{p+3;p+3}(x) = Y_{p+6;p+6}(x) = \sum_{g=1}^{8,10,11} r_{2g}\, \varsigma_g(x), Y_{ll}(x) = \sum_{g=1}^{8} r_{3g}\, \varsigma_g(x),$$
$$Y_{ij}(x) = 0, \qquad p = 1,2,3, \qquad l = 10, \ldots, 14, \qquad i, j = 1, \ldots, 14, \qquad i \neq j$$

where

$$\varsigma_g(x) = -\frac{e^{\iota\lambda_g|x|}}{4\pi|x|}, \qquad r_{1p} = \prod_{i=1, i\neq p}^{9} (\lambda_i^2 - \lambda_p^2)^{-1}, \qquad r_{2q} = \prod_{i=1, i\neq q}^{8,10,11} (\lambda_i^2 - \lambda_q^2)^{-1},$$
$$r_{3z} = \prod_{i=1, i\neq z}^{8} (\lambda_i^2 - \lambda_z^2)^{-1}, \qquad g = 1, \ldots, 11, \qquad p = 1, \ldots, 9,$$
$$q = 1, \ldots, 8, 10, 11, \qquad z = 1, \ldots, 8 \tag{55}$$

LEMMA 1: The matrix Y is the fundamental matrix of the operator $\Theta(\Delta)$ i.e.

$$\Theta(\Delta)Y(x) = \delta(x)I(x) \tag{56}$$

PROOF: To establish the lemma, it is adequate to prove that

$$\Gamma_1(\Delta)(\Delta + \lambda_9^2)Y_{11}(x) = \delta(x) \tag{57}$$

$$\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{44}(x) = \delta(x) \tag{58}$$

$$\Gamma_1(\Delta)Y_{10;10}(x) = \delta(x) \tag{59}$$

Consider $\sum_{i=1}^{9} r_{1i} = \frac{1}{z_{10}} \sum_{j=1}^{9} (-1)^{j+1} z_j$, where $z_1, \ldots, z_{10}$ are mentioned in Appendix E. Upon simplifying the R.H.S. of the above relation, we obtain

$$\sum_{i=1}^{9} r_{1i} = 0. \tag{60}$$

Similarly, we find

$$\sum_{i=2}^{9} r_{1i}(\lambda_1^2 - \lambda_i^2) = 0, \qquad \sum_{i=3}^{9} r_{1i}\left[\prod_{j=1}^{2} (\lambda_j^2 - \lambda_i^2)\right] = 0, \qquad \sum_{i=4}^{9} r_{1i}\left[\prod_{j=1}^{3} (\lambda_j^2 - \lambda_i^2)\right] = 0,$$
$$\sum_{i=5}^{9} r_{1i}\left[\prod_{j=1}^{4} (\lambda_j^2 - \lambda_i^2)\right] = 0, \qquad \sum_{i=6}^{9} r_{1i}\left[\prod_{j=1}^{5} (\lambda_j^2 - \lambda_i^2)\right] = 0, \qquad \sum_{i=7}^{9} r_{1i}\left[\prod_{j=1}^{6} (\lambda_j^2 - \lambda_i^2)\right] = 0,$$
$$\sum_{i=8}^{9} r_{1i}\left[\prod_{j=1}^{7} (\lambda_j^2 - \lambda_i^2)\right] = 0, \qquad r_{19}\left[\prod_{i=1}^{8} (\lambda_i^2 - \lambda_9^2)\right] = 1 \tag{61}$$

Also, we have

$$(\Delta+\lambda_p^2)\varsigma_g(\mathrm{x}) = \delta(\mathrm{x}) + (\lambda_p^2-\lambda_g^2)\varsigma_g(\mathrm{x}), \qquad \mathrm{p},\mathrm{g} = 1,\ldots\ldots,11 \tag{62}$$

Now, let us consider

$$\Gamma_1(\Delta)(\Delta+\lambda_9^2)\mathrm{Y}_{11}(\mathrm{x}) = \prod_{i=1}^{9}(\Delta+\lambda_i^2)\sum_{g=1}^{9} r_{1g}\varsigma_g(\mathrm{x}) = \prod_{i=2}^{9}(\Delta+\lambda_i^2)\sum_{g=1}^{9} r_{1g}\left[\delta(\mathrm{x}) + (\lambda_1^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]$$

Using Eqs. 60-62 in the above relation, we obtain

$$\begin{aligned}
&\Gamma_1(\Delta)(\Delta+\lambda_9^2)\mathrm{Y}_{11}(\mathrm{x})\\
&= \prod_{i=2}^{9}(\Delta+\lambda_i^2)\sum_{g=2}^{9} r_{1g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathrm{x}) = \prod_{i=3}^{9}(\Delta+\lambda_i^2)\sum_{g=2}^{9} r_{1g}(\lambda_1^2-\lambda_g^2)\left[\delta(\mathrm{x}) + (\lambda_2^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= \prod_{i=3}^{9}(\Delta+\lambda_i^2)\sum_{g=3}^{9} r_{1g}\left[\prod_{j=1}^{2}\left(\lambda_j^2-\lambda_g^2\right)\right]\varsigma_g(\mathrm{x})\\
&= \prod_{i=4}^{9}(\Delta+\lambda_i^2)\sum_{g=3}^{9} r_{1g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathrm{x}) + (\lambda_3^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= \prod_{i=4}^{9}(\Delta+\lambda_i^2)\sum_{g=4}^{9} r_{1g}\left[\prod_{j=1}^{3}\left(\lambda_j^2-\lambda_g^2\right)\right]\varsigma_g(\mathrm{x})\\
&= \prod_{i=5}^{9}(\Delta+\lambda_i^2)\sum_{g=4}^{9} r_{1g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathrm{x}) + (\lambda_4^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= \prod_{i=5}^{9}(\Delta+\lambda_i^2)\sum_{g=5}^{9} r_{1g}\left[\prod_{j=1}^{4}\left(\lambda_j^2-\lambda_g^2\right)\right]\varsigma_g(\mathrm{x})\\
&= \prod_{i=6}^{9}(\Delta+\lambda_i^2)\sum_{g=5}^{9} r_{1g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathrm{x}) + (\lambda_5^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= \prod_{i=6}^{9}(\Delta+\lambda_i^2)\sum_{g=6}^{9} r_{1g}\left[\prod_{j=1}^{5}\left(\lambda_j^2-\lambda_g^2\right)\right]\varsigma_g(\mathrm{x})\\
&= \prod_{i=7}^{9}(\Delta+\lambda_i^2)\sum_{g=6}^{9} r_{1g}\left[\prod_{j=1}^{5}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathrm{x}) + (\lambda_6^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= \prod_{i=7}^{9}(\Delta+\lambda_i^2)\sum_{g=7}^{9} r_{1g}\left[\prod_{j=1}^{6}\left(\lambda_j^2-\lambda_g^2\right)\right]\varsigma_g(\mathrm{x})\\
&= \prod_{i=8}^{9}(\Delta+\lambda_i^2)\sum_{g=7}^{9} r_{1g}\left[\prod_{j=1}^{6}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathrm{x}) + (\lambda_7^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= \prod_{i=8}^{9}(\Delta+\lambda_i^2)\sum_{g=8}^{9} r_{1g}\left[\prod_{j=1}^{7}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathrm{x})\\
&= (\Delta+\lambda_9^2)\sum_{g=8}^{9} r_{1g}\left[\prod_{j=1}^{7}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathrm{x}) + (\lambda_8^2-\lambda_g^2)\varsigma_g(\mathrm{x})\right]\\
&= (\Delta+\lambda_9^2)\varsigma_9(\mathrm{x}) = \delta(\mathrm{x})
\end{aligned}$$

Eqs. 58 and 59 can be proved similarly.

We introduce the matrix

$$G(x) = R(D_x)Y(x) \tag{63}$$

From Eqs. 54, 56 and 63, we obtain

$$F(D_x)G(x) = F(D_x)R(D_x)Y(x) = \Theta(\Delta)Y(x) = \delta(x)I(x)$$

Therefore, $G(x)$ is a solution to Eq. 29.

THEOREM 1: If the condition in Eq. 28 is satisfied, then the matrix $G(x)$ defined by Eq. 63 is the fundamental solution of the system of Eq. 27, and each element of the matrix $G(x)$ is represented in the following form:

$$G_{gl}(x) = R_{gl}(D_x)Y_{11}(x), \quad G_{gq}(x) = R_{gq}(D_x)Y_{44}(x), \quad G_{gj}(x) = R_{gj}(D_x)Y_{10;10}(x),$$
$$g = 1,\ldots,14, \quad l = 1,2,3, \quad q = 4,\ldots,9, \quad j = 10,\ldots,14$$

5. Basic properties of matrix $G(x)$

THEOREM 2: Each column of the matrix $G(x)$ is a solution of the system of Eq. 27 at every point $x \in E^3$ except at the origin.

THEOREM 3: If the condition in Eq. 28 is satisfied, then the fundamental solution of the system

$$\tilde{\mathbf{F}}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

is the matrix $\widetilde{G}(x) = \left(\widetilde{G}_{gl}(x)\right)_{14\times 14}$, where $\widetilde{G}_{gl}(x), g, l = 1,\ldots,14$ are given in Appendix F.

6. Fundamental solution of a system of equations in equilibrium theory

If we substitute $\omega = 0$ into the system of Eq. 27, we derive the system of equations for the equilibrium theory of thermoelastic diffusion with microtemperatures, microconcentrations, and triple porosity as:

$$\begin{gathered}
\mu\Delta u + (\lambda' + \mu)\nabla(\nabla \cdot u) + \tilde{\lambda}_i \nabla\varphi_i - \vartheta_1 \nabla\theta - \vartheta_2 \nabla P = 0, \\
(k_6\Delta - k_2)v + (k_4 + k_5)\nabla(\nabla \cdot v) - k_3 \nabla\theta = 0, \\
(h_6\Delta - h_2)w + (h_4 + h_5)\nabla(\nabla \cdot w) - h_3 \nabla P = 0, \\
-\tilde{\lambda}_1(\nabla \cdot u) - B_1(\nabla \cdot v) - B_4(\nabla \cdot w) + (A_1\Delta - \zeta_1)\varphi_1 + (A_4\Delta - \zeta_4)\varphi_2 + (A_6\Delta - \zeta_6)\varphi_3 \\
+\xi_1\theta + v_1 P = 0, \\
-\tilde{\lambda}_2(\nabla \cdot u) - B_2(\nabla \cdot v) - B_5(\nabla \cdot w) + (A_4\Delta - \zeta_4)\varphi_1 + (A_2\Delta - \zeta_2)\varphi_2 + (A_5\Delta - \zeta_5)\varphi_3 \\
+\xi_2\theta + v_2 P = 0, \\
-\tilde{\lambda}_3(\nabla \cdot u) - B_3(\nabla \cdot v) - B_6(\nabla \cdot w) + (A_6\Delta - \zeta_6)\varphi_1 + (A_5\Delta - \zeta_5)\varphi_2 + (A_3\Delta - \zeta_3)\varphi_3 \\
+\xi_3\theta + v_3 P = 0, \\
k_1(\nabla \cdot v) + k\Delta\theta = 0, \\
h_1(\nabla \cdot w) + h\Delta P = 0
\end{gathered} \tag{64}$$

We introduce the second-order matrix differential operators with constant coefficients as

$$\mathbf{E}(\mathbf{D}_x) = \left(\mathbf{E}_{gl}(\mathbf{D}_x)\right)_{14\times 14}$$

where matrix $E(D_x)$ can be obtained from $F(D_x)$ by taking $\omega = 0$. The system of Eq. 64 can be represented as

$$\mathbf{E}(\mathbf{D}_x)\mathbf{U}(\mathbf{x}) = \mathbf{0}$$

DEFINITION 3: The operator $E(D_x)$ is an elliptic differential operator iff Eq. 28 is satisfied.

DEFINITION 4: The fundamental solution of the system of Eq. 64 is the matrix $G'(x) = \left(G'_{gl}(x)\right)_{14\times14}$ satisfying the following condition.

$$E(D_x)G'(x) = \delta(x)I(x) \tag{65}$$

We consider the system of non-homogeneous equations

$$\mu\Delta u + (\lambda' + \mu)\nabla(\nabla\cdot u) - \tilde{\lambda}_i\nabla\varphi_i = H' \tag{66}$$

$$(k_6\Delta - k_2)v + (k_4 + k_5)\nabla(\nabla\cdot v) - B_i\nabla\varphi_i + k_1\nabla\theta = V' \tag{67}$$

$$(h_6\Delta - h_2)w + (h_4 + h_5)\nabla(\nabla\cdot w) - B_{i+3}\nabla\varphi_i + h_1P = W' \tag{68}$$

$$\tilde{\lambda}_1(\nabla\cdot u) + (A_1\Delta - \zeta_1)\varphi_1 + (A_4\Delta - \zeta_4)\varphi_2 + (A_6\Delta - \zeta_6)\varphi_3 = X'_1, \tag{69}$$

$$\tilde{\lambda}_2(\nabla\cdot u) + (A_4\Delta - \zeta_4)\varphi_1 + (A_2\Delta - \zeta_2)\varphi_2 + (A_5\Delta - \zeta_5)\varphi_3 = X'_2, \tag{70}$$

$$\tilde{\lambda}_3(\nabla\cdot u) + (A_6\Delta - \zeta_6)\varphi_1 + (A_5\Delta - \zeta_5)\varphi_2 + (A_3\Delta - \zeta_3)\varphi_3 = X'_3, \tag{71}$$

$$-\vartheta_1(\nabla\cdot u) - k_3(\nabla\cdot v) + \xi_i\varphi_i + k\Delta\theta = Y' \tag{72}$$

$$-\vartheta_2(\nabla\cdot u) - h_3(\nabla\cdot w) + v_i\varphi_i + h\Delta P = Z' \tag{73}$$

where $\mathbf{H}', \mathbf{V}', \mathbf{W}'$ are three-component vector functions on E^3; $\mathbf{X}'_i, \mathbf{Y}', \mathbf{Z}'$ are scalar functions on E^3. The Eqs. 66–73 can also be expressed in the following form:

$$E^{tr}(D_x)U(x) = Q'(x) \tag{74}$$

where E^{tr} is the transpose of matrix $E, Q' = (H', V', W', X'_i, Y', Z')$, $x \in E^3$. Applying the operator $(\nabla\,\cdot)$ to the Eqs. 66-68, we obtain

$$\tilde{\lambda}\Delta(\nabla\cdot u) - \tilde{\lambda}_i\Delta\varphi_i = \nabla\cdot H' \tag{75}$$

$$(k_7\Delta - k_2)\nabla\cdot v - B_i\Delta\varphi_i + k_1\Delta\theta = \nabla\cdot V' \tag{76}$$

$$(h_7\Delta - h_2)\nabla\cdot w - B_{i+3}\Delta\varphi_i + h_1\Delta P = \nabla\cdot W' \tag{77}$$

Eqs. 69-71 and 75 may be expressed in the form

$$N'(\Delta) \rightleftarrows\rightleftarrows S' = \tilde{Q}' \tag{78}$$

where $S', \tilde{Q}'$ and $N'(\Delta)$ are defined in Appendix G. Eq. 78 can also be expressed in determinant form as:

$$\Gamma_3(\Delta)S' = \Phi \tag{79}$$

where $\Phi = (\Phi_1, \ldots, \Phi_4), \Phi_p = \frac{1}{\tilde{C}}\sum_{i=1}^{4} M^*_{ip}w_i, \Gamma_3(\Delta) = \frac{|N'(\Delta)|}{\tilde{C}}, \tilde{C} = \tilde{\lambda}\vartheta, p = 1, \ldots, 4$ and M^*_{ip} is the cofactor of the element N'_{ip} of the matrix N'. On expanding $\Gamma_3(\Delta)$, we see that

$$\Gamma_3(\Delta) = \Delta \prod_{3}^{i=1} (\Delta + \tilde{\mu}_i^2)$$

where $\tilde{\mu}_i^2, i = 1,2,3$ are the roots of the equation $\left|\hat{N}(-m)\right| = 0$ (with respect to m). From Eq. 72, we get $\Delta\theta = \frac{1}{k}[Y' + \vartheta_1(\nabla\cdot u) + k_3(\nabla\cdot v) - \xi_i\varphi_i]$. Use it in Eq. 76 and then apply the operator $\Gamma_3(\Delta)$ to the resulting equation, we get

$$\Gamma_3(\Delta)(\Delta+\tilde{\mu}_4^2)\nabla\cdot v=\Phi_5,\tilde{\mu}_4^2=\frac{k_1k_3-k_2k}{kk_7} \tag{80}$$

where Φ_5 is given in Appendix G. From Eq. 73, we get $\Delta P=\frac{1}{h}[Z'+\vartheta_2(\nabla\cdot u)+h_3(\nabla\cdot w)-v_i\varphi_i]$. Use it in Eq. 77 and then apply the operator $\Gamma_3(\Delta)$ to the resulting equation, we get

$$\Gamma_3(\Delta)(\Delta+\tilde{\mu}_5^2)\nabla\cdot w=\Phi_6,\tilde{\mu}_5^2=\frac{h_1h_3-h_2h}{hh_7} \tag{81}$$

where Φ_6 is given in Appendix G. Applying operators $\Gamma_3(\Delta)(\Delta+\tilde{\mu}_4^2)$ and $\Gamma_3(\Delta)(\Delta+\tilde{\mu}_5^2)$ to Eqs. 72 and 73 and using Eqs. 80 and 81, we get

$$\Delta^2\prod_{i=1}^{4}(\Delta+\tilde{\mu}_i^2)\theta=\Phi_7 \tag{82}$$

$$\Delta^2\prod_{i=1}^{3,5}(\Delta+\tilde{\mu}_i^2)P=\Phi_8 \tag{83}$$

where Φ_7 and Φ_8 are defined in Appendix G. Applying operators $\Gamma_3(\Delta)$, $\Delta^2\prod_{i=1}^{4}(\Delta+\tilde{\mu}_i^2)$ and $\Delta^2\prod_{i=1}^{3,5}(\Delta+\tilde{\mu}_i^2)$ to Eqs. 66-68 respectively and using Eqs. 79-83, we obtain

$$\begin{aligned}&\Delta\Gamma_3(\Delta)u=\Phi',\\&\Delta^2\prod_{i=1}^{4,6}(\Delta+\tilde{\mu}_i^2)v=\Phi'',\qquad \tilde{\mu}_6^2=-\frac{k_2}{k_6}\\&\Delta^2\prod_{i=1}^{3,5,7}(\Delta+\tilde{\mu}_i^2)w=\Phi''',\qquad \tilde{\mu}_7^2=-\frac{h_2}{h_6}\end{aligned} \tag{84}$$

where Φ',Φ'',Φ''' are mentioned in Appendix G. From Eqs. 79 and 82-84, we get

$$\Lambda(\Delta)U(x)=\hat{\Phi}(x) \tag{85}$$

where $\hat{\Phi}(x)$ and $\Lambda(\Delta)$ are defined in appendix G.

The expressions for $\Phi',\Phi'',\Phi''',\Phi_2,\Phi_3,\Phi_4,\Phi_7,\Phi_8$ can be rewritten as

$$\widehat{\Phi}(x)=Z^{tr}(D_x)Q'(x) \tag{86}$$

where $Z(D_x)=\left(Z_{gl}(D_x)\right)_{14\times14}$ is given in Appendix H. From Eqs. 74, 85, and 86, we get

$$E(D_x)R(D_x)=\Lambda(\Delta) \tag{87}$$

Let

$$Y'(x)=\left(Y'_{ij}(x)\right)_{14\times14},\qquad Y'_{pp}(x)=r'_{11}\varsigma_1^*(x)+r'_{12}\varsigma_2^*(x)+\sum_{g=1}^{3}r'_{1;g+2}\,\hat{\varsigma}_g(x),$$

$$Y'_{p+3;p+3}(x)=r'_{21}\varsigma_1^*(x)+r'_{22}\varsigma_2^*(x)+\sum_{g=1}^{4,6}r'_{2;g+2}\,\hat{\varsigma}_g(x),$$

$$Y'_{p+6;p+6}(x)=r'_{31}\varsigma_1^*(x)+r'_{32}\varsigma_2^*(x)+\sum_{g=1}^{3,5,7}r'_{3;g+2}\,\hat{\varsigma}_g(x),$$

$$Y'_{p+9;p+9}(x) = r'_{41}\varsigma_1^*(x) + \sum_{g=1}^{3} r'_{4;g+1}\,\hat{\varsigma}_g(x),$$

$$Y'_{13;13}(x) = r'_{51}\varsigma_1^*(x) + r'_{52}\varsigma_2^*(x) + \sum_{g=1}^{4} r'_{5;g+2}\,\hat{\varsigma}_g(x),$$

$$Y'_{14;14}(x) = r'_{61}\varsigma_1^*(x) + r'_{62}\varsigma_2^*(x) + \sum_{g=1}^{3,5} r'_{6;g+2}\,\hat{\varsigma}_g(x),$$

$$Y'_{ij}(x) = 0, \qquad p = 1,2,3, \qquad i,j = 1,\ldots.,14, \qquad i \neq j$$

where $\hat{\varsigma}_g(x), g = 1,\ldots,7$ and $r'_{11},\ldots,r'_{67}$ are given in Appendix I.

LEMMA 2: The matrix $Y'$is the fundamental matrix of the operator $\Lambda(\Delta)$ i.e.

$$\Lambda(\Delta)Y'(x) = \delta(x)I(x) \tag{88}$$

PROOF: To prove the lemma, it is sufficient to prove that

$$\begin{gathered}\Delta\Gamma_3(\Delta)Y'_{11}(x) = \delta(x),\\ \Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_4^2)(\Delta+\tilde{\mu}_6^2)Y'_{44}(x) = \delta(x),\\ \Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_5^2)(\Delta+\tilde{\mu}_7^2)Y'_{77}(x) = \delta(x),\\ \Gamma_3(\Delta)Y'_{10;10}(x) = \delta(x),\\ \Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_4^2)Y'_{13;13}(x) = \delta(x),\\ \Delta\Gamma_3(\Delta)(\Delta+\tilde{\mu}_5^2)Y'_{14;14}(x) = \delta(x)\end{gathered} \tag{89}$$

It is much easier to prove the system of Eq. 89. It has been left for the reader. We introduce the matrix

$$G'(x) = Z(D_x)Y'(x) \tag{90}$$

From Eqs. 87, 88 and 90, we obtain $E(D_x)G'(x) = \delta(x)I(x)$. Hence, $G(x)$ is a solution to Eq. 65.

THEOREM 4: If the condition in Eq. 29 is satisfied, then the matrix $G'(x)$ defined by Eq. 90 is the fundamental solution of the system of Eq. 64.

7. Conclusions

The paper makes significant contributions to the field of thermoelasticity through the following key aspects:

- **Derivation of linear theory:** The paper introduces a linear theory that incorporates triple porosity, microtemperatures, and microconcentrations. This theoretical framework expands the understanding of thermoelastic behavior in materials by considering multiple intricate factors simultaneously.
- **Analysis of fundamental matrix:** The study includes an analysis of the fundamental matrix associated with the resulting system of equations. This analysis is conducted under different scenarios, including oscillatory and equilibrium conditions. Understanding the fundamental matrix provides insights into the dynamic and static behavior of the thermoelastic system.
- **Enhanced understanding of thermoelastic behavior:** By considering triple porosity, microtemperatures, and microconcentrations in the derivation of the linear theory, the paper contributes to a deeper and more nuanced understanding of thermoelastic behavior in complex materials. This holistic approach enables researchers and practitioners to model and analyze materials with a higher degree of accuracy.
- **Practical insights:** The derived linear theory and the analysis of the fundamental matrix offer practical insights that can be applied in real-world scenarios. Understanding how thermoelastic

materials behave under various conditions is crucial for designing and optimizing materials in engineering and scientific applications.

In summary, the paper's contributions provide a valuable foundation for advancing the field of thermoelasticity, offering a more comprehensive theoretical framework and insights that can be practically applied in the analysis and design of materials with complex thermoelastic properties.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

No funding was received for this study.

Data availability statement

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

References

[1] Svanadze M (2016) Fundamental solutions in the theory of elasticity for triple porosity materials. Meccanica 51:1825–1837.
[2] Straughan B (2017) Uniqueness and stability in triple porosity thermoelasticity. Rendiconti Lincei - Matematica e Applicazioni 28:191–208.
[3] Kansal T (2023) Fundamental solutions in generalized theory of thermoelastic diffusion with triple porosity. Engineering Transactions 71:473–505.
[4] Svanadze M (2016) External boundary value problems in the quasi static theory of elasticity for triple porosity materials. PAMM 16:495–496.
[5] Svanadze M (2016) Boundary value problems in the theory of thermoelasticity for triple porosity materials. Mechanics of Solids, Structures and Fluids; NDE, Diagnosis, and Prognosis 9:1–10.
[6] Svanadze M (2017) External boundary value problems in the quasi static theory of triple porosity thermoelasticity. PAMM 17:471–472.
[7] Svanadze M (2018) Potential method in the theory of elasticity for triple porosity materials. Journal of Elasticity 130:1–24.
[8] Svanadze M (2018) Potential method in the linear theory of triple porosity thermoelasticity. Journal of Mathematical Analysis and Applications 461:1585–1605.
[9] Svanadze M (2018) On the linear equilibrium theory of elasticity for materials with triple voids. The Quarterly Journal of Mechanics and Applied Mathematics 71:329–348.
[10] Straughan B (2017) Mathematical aspects of multi-porosity continua. Advances in Mechanics and Mathematics 38:1–208.
[11] Grot RA (1969) Thermodynamics of a continuum with microstructure. International Journal of Engineering Science 7:801–814.
[12] Iesan D, Quintanilla R (2000) On a theory of thermoelasticity with microtemperatures. Journal of Thermal Stresses 23:199–215.
[13] Aouadi M, Ciarletta M, Tibullo V (2017) A thermoelastic diffusion theory with microtemperatures and microconcentrations. Journal of Thermal Stresses 40:486–501.
[14] Chirilă A, Marin M (2019) Diffusion in microstretch thermoelasticity with microtemperatures and microconcentrations. In: Models and Theories in Social Systems, pp. 149–164.
[15] Svanadze M (2004) Fundamental solutions of the equations of the theory of thermoelasticity with microtemperatures. Journal of Thermal Stresses 27:151–170.

[16] Svanadze M (2004) Fundamental solution in the theory of micromorphic elastic solids with microtemperatures. Journal of Thermal Stresses 27:345–366.
[17] Svanadze M (2019) Fundamental solutions in the linear theory of thermoelasticity for solids with triple porosity. Mathematics and Mechanics of Solids 24:919–938.

Appendix A

$$F_{pq}(D_x) = [\mu\Delta + \rho\omega^2]\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p\,\partial x_q}, \qquad F_{p;q+3}(D_x) = F_{p+3;q}(D_x) = 0,$$

$$F_{p;q+6}(D_x) = F_{p+6;q}(D_x) = 0, \qquad F_{p;q+9}(D_x) = -F_{q+9;p}(D_x) = \tilde{\lambda}_q\frac{\partial}{\partial x_p},$$

$$F_{p;13}(D_x) = -\vartheta_1\frac{\partial}{\partial x_p}, F_{p;14}(D_x) = -\vartheta_2\frac{\partial}{\partial x_p},$$

$$F_{p+3;q+3}(D_x) = [k_6\Delta + k_8]\delta_{pq} + (k_4 + k_5)\frac{\partial^2}{\partial x_p\,\partial x_q},$$

$$F_{p+3;q+6}(D_x) = F_{p+6;q+3}(D_x) = \iota\omega D_3\delta_{pq}, F_{p+3;q+9}(D_x) = \iota\omega B_q\frac{\partial}{\partial x_p},$$

$$F_{p+3;13}(D_x) = -k_3\frac{\partial}{\partial x_p}, F_{p+3;14}(D_x) = F_{14;p+3}(D_x) = 0,$$

$$F_{p+6;q+6}(D_x) = [h_6\Delta + h_8]\delta_{pq} + (h_4 + h_5)\frac{\partial^2}{\partial x_p\,\partial x_q},$$

$$F_{p+6;q+9}(D_x) = \iota\omega B_{q+3}\frac{\partial}{\partial x_p}, F_{p+6;13}(D_x) = F_{13;p+6}(D_x) = 0, F_{p+6;14}(D_x) = -h_3\frac{\partial}{\partial x_p},$$

$$F_{p+9;q+3}(D_x) = -B_p\frac{\partial}{\partial x_q}, F_{p+9;q+6}(D_x) = -B_{p+3}\frac{\partial}{\partial x_q}, F_{p+9;p+9}(D_x) = A_p\Delta - \gamma_p,$$

$$F_{10;11}(D_x) = F_{11;10}(D_x) = A_4\Delta - \zeta_4, F_{10;12}(D_x) = F_{12;10}(D_x) = A_6\Delta - \zeta_6,$$

$$F_{11;12}(D_x) = F_{12;11}(D_x) = A_5\Delta - \zeta_5, F_{p+9;13}(D_x) = \xi_p, F_{p+9;14}(D_x) = v_p,$$

$$F_{13;q}(D_x) = \iota\omega\vartheta_1 T_0\frac{\partial}{\partial x_q}, F_{13;q+3}(D_x) = k_1\frac{\partial}{\partial x_q}, F_{13;q+9}(D_x) = \iota\omega\xi_q T_0,$$

$$F_{13;13}(D_x) = k\Delta + \iota\omega\eta T_0, F_{13;14}(D_x) = \iota\omega\varsigma T_0, F_{14;q}(D_x) = \iota\omega\vartheta_2\frac{\partial}{\partial x_q},$$

$$F_{14;q+6}(D_x) = h_1\frac{\partial}{\partial x_q}, F_{14;q+9}(D_x) = \iota\omega v_q, F_{14;13}(D_x) = \iota\omega\varsigma,$$

$$F_{14;14}(D_x) = h\Delta + \iota\omega\varpi, p,q = 1,2,3,\; k_8 = \iota\omega D_1 - k_2,\; h_8 = \iota\omega D_2 - h_2$$

$$\tilde{F}_{pq}(D_x) = \mu\Delta\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p\,\partial x_q}, \qquad \tilde{F}_{p+3;q+3}(D_x) = k_6\Delta\delta_{pq} + (k_4 + k_5)\frac{\partial^2}{\partial x_p\,\partial x_q},$$

$$\tilde{F}_{p+6;q+6}(D_x) = h_6\Delta\delta_{pq} + (h_4 + h_5)\frac{\partial^2}{\partial x_p\,\partial x_q}, \qquad \tilde{F}_{p+6;p+6}(D_x) = A_p\Delta, \tilde{F}_{10;11}(D_x) =$$

$$\tilde{F}_{11;10}(D_x) = A_4\Delta, \qquad \tilde{F}_{10;12}(D_x) = \tilde{F}_{12;10}(D_x) = A_6\Delta, \qquad \tilde{F}_{11;12}(D_x) = \tilde{F}_{12;11}(D_x) = A_5\Delta,$$

$$\tilde{F}_{13;13}(D_x) = k\Delta, \qquad \tilde{F}_{14;14}(D_x) = h\Delta, \qquad \tilde{F}_{pz}(D_x) = \tilde{F}_{zp}(D_x) = 0,$$

$$\tilde{F}_{p+3;y}(D_x) = \tilde{F}_{y;p+3}(D_x) = 0, \qquad \tilde{F}_{p+6;i}(D_x) = \tilde{F}_{i;p+6}(D_x) = 0, \qquad \tilde{F}_{p+9;j}(D_x) = \tilde{F}_{j;p+9}(D_x) = 0,$$

$$\tilde{F}_{13;14}(D_x) = \tilde{F}_{14;13}(D_x) = 0, p,q = 1,2,3, z = 4,\ldots,14, y = 7,\ldots,14, i = 10,\ldots,14, j = 13,14$$

Appendix B

$$S = (\nabla\cdot u, \nabla\cdot v, \nabla\cdot w, \varphi, \theta, P), \qquad \widetilde{Q} = (w_1, \ldots, w_8) = (\nabla\cdot H, \nabla\cdot V, \nabla\cdot W, X_i, Y, Z)$$

$$N(\Delta) = \left(N_{gl}(\Delta)\right)_{8\times 8} = \begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 0 0 - \tilde{\lambda}_1\Delta - \tilde{\lambda}_2\Delta - \tilde{\lambda}_3\Delta \iota\omega T_0\vartheta_1\Delta\iota\omega\vartheta_2\Delta \\ 0 k_7\Delta + k_8\iota\omega D_3 - B_1\Delta - B_2\Delta - B_3\Delta k_1\Delta 0 \\ 0\iota\omega D_3 h_7\Delta + h_8 - B_4\Delta - B_5\Delta - B_6\Delta 0 h_1\Delta \\ \tilde{\lambda}_1\iota\omega B_1\iota\omega B_4 A_1\Delta - \gamma_1 A_4\Delta - \zeta_4 A_6\Delta - \zeta_6\iota\omega T_0\xi_1\iota\omega v_1 \\ \tilde{\lambda}_2\iota\omega B_2\iota\omega B_5 A_4\Delta - \zeta_4 A_2\Delta - \gamma_2 A_5\Delta - \zeta_5\iota\omega T_0\xi_2\iota\omega v_2 \\ \tilde{\lambda}_3\iota\omega B_3\iota\omega B_6 A_6\Delta - \zeta_6 A_5\Delta - \zeta_5 A_3\Delta - \gamma_3\iota\omega T_0\xi_3\iota\omega v_3 \\ -\vartheta_1 - k_3 0\xi_1\xi_2\xi_3 k\Delta + \iota\omega T_0\eta\iota\omega\varsigma \\ -\vartheta_2 0 - h_3 v_1 v_2 v_3\iota\omega T_0\varsigma h\Delta + \iota\omega\varpi \end{pmatrix}_{8\times 8}$$

Appendix C

$$\lambda_9^2 = \frac{\rho\omega^2}{\mu}, \qquad \Psi' = \frac{1}{\mu}\{\Gamma_1(\Delta)H\text{-}\nabla[(\lambda' + \mu)\psi_1 - \tilde{\lambda}_i\psi_{i+3} + \iota\omega\vartheta_1 T_0\psi_7 + \iota\omega\vartheta_2\psi_8]\}$$

$$\Gamma_2(\Delta) = \frac{1}{\tilde{\tilde{B}}}\begin{vmatrix} k_6\Delta + k_8\iota\omega D_3 \\ \iota\omega D_3 h_6\Delta + h_8 \end{vmatrix}, \qquad \tilde{\tilde{B}} = k_6 h_6$$

$$\Psi'' = \frac{1}{\tilde{\tilde{B}}}\begin{Bmatrix} (h_6\Delta + h_8)[\Gamma_1(\Delta)V - (k_4 + k_5)\nabla\psi_2 + B_i\nabla\psi_{i+3} - k_1\nabla\psi_7] \\ -\iota\omega D_3[\Gamma_1(\Delta)W - (h_4 + h_5)\nabla\psi_3 + B_{i+3}\nabla\psi_{i+3} - h_1\nabla\psi_8] \end{Bmatrix}$$

$$\Psi''' = \frac{1}{\tilde{\tilde{B}}}\begin{Bmatrix} (k_6\Delta + k_8)[\Gamma_1(\Delta)W - (h_4 + h_5)\nabla\psi_3 + B_{i+3}\nabla\psi_{i+3} - h_1\nabla\psi_8] \\ -\iota\omega D_3[\Gamma_1(\Delta)V - (k_4 + k_5)\nabla\psi_2 + B_i\nabla\psi_{i+3} - k_1\nabla\psi_7] \end{Bmatrix}$$

$$\widehat{\Psi}(x) = (\Psi', \Psi'', \Psi''', \Psi_4, \ldots, \Psi_8), \quad \Theta(\Delta) = \left(\Theta_{gl}(\Delta)\right)_{14\times 14},$$

$$\Theta_{pp}(\Delta) = \Gamma_1(\Delta)(\Delta + \lambda_9^2) = \prod_{9}^{i=1}(\Delta + \lambda_i^2), \qquad \Theta_{p+3;p+3}(\Delta) = \Theta_{p+6;p+6}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{8,10,11}^{i=1}(\Delta + \lambda_i^2),$$

$$\Theta_{qq}(\Delta) = \Gamma_1(\Delta) = \prod_{8}^{i=1}(\Delta + \lambda_i^2), \Theta_{gl}(\Delta) = 0, p = 1,2,3, q = 10,\ldots,14, g, l = 1,\ldots,14, g \neq l$$

Appendix D

$$w_{p1}(\Delta) = -\frac{1}{\tilde{A}\mu}\left[(\lambda' + \mu)N_{p1}^*(\Delta) - [\tilde{\lambda}_i N_{p;i+3}^*(\Delta) - \iota\omega\vartheta_1 T_0 N_{p7}^*(\Delta) - \iota\omega\vartheta_2 N_{p8}^*(\Delta)]\right],$$

$$w_{p2}(\Delta) = -\frac{1}{\tilde{A}\tilde{\tilde{B}}}\begin{Bmatrix} (h_6\Delta + h_8)[(k_4 + k_5)N_{p2}^*(\Delta) - B_i N_{p;i+3}^*(\Delta) + k_1 N_{p7}^*(\Delta)] \\ -\iota\omega D_3[(h_4 + h_5)N_{p3}^*(\Delta) - B_{i+3}N_{p;i+3}^*(\Delta) + h_1 N_{p8}^*(\Delta)] \end{Bmatrix},$$

$$w_{p3}(\Delta) = -\frac{1}{\tilde{A}\tilde{\tilde{B}}}\begin{Bmatrix} -\iota\omega D_3[(k_4 + k_5)N_{p2}^*(\Delta) - B_i N_{p;i+3}^*(\Delta) + k_1 N_{p7}^*(\Delta)] + \\ (k_6\Delta + k_8)[(h_4 + h_5)N_{p3}^*(\Delta) - B_{i+3}N_{p;i+3}^*(\Delta) + h_1 N_{p8}^*(\Delta)] \end{Bmatrix},$$

$$w_{pj}(\Delta) = \frac{N_{pj}^*(\Delta)}{\tilde{A}}, p = 1,\ldots,8, j = 4,\ldots,8$$

$$R_{ij}(D_x) = \frac{1}{\mu}\Gamma_1(\Delta)\delta_{ij} + w_{11}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j}, \qquad R_{i;j+3}(D_x) = w_{12}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

$$R_{i;j+6}(D_x) = w_{13}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j}, \qquad R_{i;p+6}(D_x) = w_{1p}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{i+3;j}(D_x) = w_{21}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j}, \qquad R_{i+3;j+3}(D_x) = \frac{1}{\tilde{\tilde{B}}}(h_6\Delta + h_8)\Gamma_1(\Delta)\delta_{ij} + w_{22}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

$$R_{i+3;j+6}(D_x) = -\frac{1}{\tilde{\tilde{B}}}\iota\omega D_3\Gamma_1(\Delta)\delta_{ij} + w_{23}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j}, \qquad R_{i+3;p+6}(D_x) = w_{2p}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{i+6;j}(D_x)=w_{31}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},\qquad R_{i+6;j+3}(D_x)=-\frac{1}{\tilde{\tilde{B}}}\iota\omega D_3\Gamma_1(\Delta)\delta_{ij}+w_{32}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

$$R_{i+6;j+6}(D_x)=\frac{1}{\tilde{\tilde{B}}}(k_6\Delta+k_8)\Gamma_1(\Delta)\delta_{ij}+w_{33}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},\qquad R_{i+6;p+6}(D_x)=w_{3p}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{p+6;i}(D_x)=w_{p1}(\Delta)\frac{\partial}{\partial x_i},\qquad R_{p+6;i+3}(D_x)=w_{p2}(\Delta)\frac{\partial}{\partial x_i},\qquad R_{p+6;i+6}(D_x)=w_{p3}(\Delta)\frac{\partial}{\partial x_i},$$

$$R_{p+6;e+6}(D_x)=w_{pe}(\Delta),\ i,j=1,2,3,\ p,e=4,\ldots,8$$

Appendix E

$$z_1=\prod_{i=3}^{9}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{9}(\lambda_3^2-\lambda_j^2)\prod_{p=5}^{9}(\lambda_4^2-\lambda_p^2)\prod_{q=6}^{9}(\lambda_5^2-\lambda_q^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_7^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_2=\prod_{i=3}^{9}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{9}(\lambda_3^2-\lambda_j^2)\prod_{p=5}^{9}(\lambda_4^2-\lambda_p^2)\prod_{q=6}^{9}(\lambda_5^2-\lambda_q^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_7^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_3=\prod_{\substack{i=2\\ i\neq 3}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{9}(\lambda_2^2-\lambda_j^2)\prod_{p=5}^{9}(\lambda_4^2-\lambda_p^2)\prod_{q=6}^{9}(\lambda_5^2-\lambda_q^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_7^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_4=\prod_{\substack{i=2\\ i\neq 4}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 4}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{p=5}^{9}(\lambda_3^2-\lambda_p^2)\prod_{q=6}^{9}(\lambda_5^2-\lambda_q^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_7^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_5=\prod_{\substack{i=2\\ i\neq 5}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 5}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{\substack{p=4\\ p\neq 5}}^{9}(\lambda_3^2-\lambda_p^2)\prod_{q=6}^{9}(\lambda_4^2-\lambda_q^2)\prod_{g=7}^{9}(\lambda_6^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_7^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_6=\prod_{\substack{i=2\\ i\neq 6}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 6}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{\substack{p=4\\ p\neq 6}}^{9}(\lambda_3^2-\lambda_p^2)\prod_{\substack{q=5\\ q\neq 6}}^{9}(\lambda_4^2-\lambda_q^2)\prod_{g=7}^{9}(\lambda_5^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_7^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_7=\prod_{\substack{i=2\\ i\neq 7}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 7}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{\substack{p=4\\ p\neq 7}}^{9}(\lambda_3^2-\lambda_p^2)\prod_{\substack{q=5\\ q\neq 7}}^{9}(\lambda_4^2-\lambda_q^2)\prod_{\substack{g=6\\ g\neq 7}}^{9}(\lambda_5^2-\lambda_g^2)\prod_{l=8}^{9}(\lambda_6^2-\lambda_l^2)(\lambda_8^2-\lambda_9^2),$$

$$z_8=\prod_{\substack{i=2\\ i\neq 8}}^{9}(\lambda_1^2-\lambda_i^2)\prod_{\substack{j=3\\ j\neq 8}}^{9}(\lambda_2^2-\lambda_j^2)\prod_{\substack{p=4\\ p\neq 8}}^{9}(\lambda_3^2-\lambda_p^2)\prod_{\substack{q=5\\ q\neq 8}}^{9}(\lambda_4^2-\lambda_q^2)\prod_{\substack{g=6\\ g\neq 8}}^{9}(\lambda_5^2-\lambda_g^2)\prod_{\substack{l=7\\ l\neq 8}}^{9}(\lambda_6^2-\lambda_l^2)(\lambda_7^2-\lambda_9^2),$$

$$z_9=\prod_{i=2}^{8}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{8}(\lambda_2^2-\lambda_j^2)\prod_{p=4}^{8}(\lambda_3^2-\lambda_p^2)\prod_{q=5}^{8}(\lambda_4^2-\lambda_q^2)\prod_{g=6}^{8}(\lambda_5^2-\lambda_g^2)\prod_{l=7}^{8}(\lambda_6^2-\lambda_l^2)(\lambda_7^2-\lambda_8^2),$$

$$z_{10}=\prod_{i=2}^{9}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{9}(\lambda_2^2-\lambda_j^2)\prod_{p=4}^{9}(\lambda_3^2-\lambda_p^2)\prod_{q=5}^{9}(\lambda_4^2-\lambda_q^2)\prod_{g=6}^{9}(\lambda_5^2-\lambda_g^2)\prod_{l=7}^{9}(\lambda_6^2-\lambda_l^2)\prod_{e=8}^{9}(\lambda_7^2-\lambda_e^2)(\lambda_8^2-\lambda_9^2)$$

Appendix F

$$\tilde{G}_{ij}(x)=\frac{1}{\tilde{\lambda}}\nabla(\nabla\cdot\varsigma_2^*(x))-\frac{1}{\mu}\nabla\times(\nabla\times\varsigma_2^*(x)),\qquad \tilde{G}_{i+3;j+3}(x)=\frac{1}{k_7}\nabla(\nabla\cdot\varsigma_2^*(x))-\frac{1}{k_6}\nabla\times(\nabla\times\varsigma_2^*(x)),$$

$$\tilde{G}_{i+6;j+6}(x)=\frac{1}{h_7}\nabla(\nabla\cdot\varsigma_2^*(x))-\frac{1}{h_6}\nabla\times(\nabla\times\varsigma_2^*(x)),\qquad \tilde{G}_{10;10}(x)=\frac{A_2A_3-A_5^2}{\vartheta}\varsigma_1^*(x),$$

$$\tilde{G}_{10;11}(x)=\tilde{G}_{11;10}(x)=\frac{A_5A_6-A_4A_3}{\vartheta}\varsigma_1^*(x),\qquad \tilde{G}_{10;12}(x)=\tilde{G}_{12;10}(x)=\frac{A_4A_5-A_2A_6}{\vartheta}\varsigma_1^*(x),$$

$$\tilde{G}_{11;11}(x)=\frac{A_1A_3-A_6^2}{\vartheta}\varsigma_1^*(x),\qquad \tilde{G}_{11;12}(x)=\tilde{G}_{12;11}(x)=\frac{A_4A_6-A_1A_5}{\vartheta}\varsigma_1^*(x),$$

$$\tilde{G}_{12;12}(x)=\frac{A_1A_2-A_4^2}{\vartheta}\varsigma_1^*(x),\qquad \tilde{G}_{13;13}(x)=\frac{\varsigma_1^*(x)}{k},\qquad \tilde{G}_{14;14}(x)=\frac{\varsigma_1^*(x)}{h},$$

$$\widetilde{G}_{ip}(x) = \widetilde{G}_{pi}(x) = \widetilde{G}_{i+3;q}(x) = \widetilde{G}_{q;i+3}(x) = \widetilde{G}_{i+6;y}(x) = \widetilde{G}_{y;i+6}(x) = \widetilde{G}_{i+9;r}(x) = 0,$$

$$\widetilde{G}_{r;i+9}(x) = \widetilde{G}_{13;14}(x) = \widetilde{G}_{14;13}(x) = 0, \qquad \varsigma_1^*(x) = -\frac{1}{4\pi|x|}, \qquad \varsigma_2^*(x) = -\frac{|x|}{8\pi}$$

$i, j = 1,2,3,\ p = 4,\dots,14,\ q = 7,\dots,14,\ y = 10,\dots,14,\ r = 13,14$

Appendix G

$$S' = (\nabla \cdot u, \varphi), \widetilde{Q}' = (w_1', w_2', w_3', w_4') = (\nabla \cdot H', X_i')$$

$$\hat{N}(\Delta) = \left(\hat{N}_{gl}(\Delta)\right)_{4\times4} = \begin{pmatrix} \lambda - \tilde{\lambda}_1 - \tilde{\lambda}_2 - \tilde{\lambda}_3 \\ \tilde{\lambda}_1 A_1\Delta - \zeta_1 A_4\Delta - \zeta_4 A_6\Delta - \zeta_6 \\ \tilde{\lambda}_2 A_4\Delta - \zeta_4 A_2\Delta - \zeta_2 A_5\Delta - \zeta_5 \\ \tilde{\lambda}_3 A_6\Delta - \zeta_6 A_5\Delta - \zeta_5 A_3\Delta - \zeta_3 \end{pmatrix}_{4\times4},$$

$$N'(\Delta) = \left(N'_{gl}(\Delta)\right)_{4\times4} = \Delta\left(\hat{N}_{gh}(\Delta)\right)_{4\times4}$$

$$\Phi_5 = \frac{1}{kk_7}\left\{k\left[\left(B_i\Delta + \frac{\xi_i k_1}{k}\right)\Phi_{i+1} + \Gamma_3(\Delta)\nabla\cdot V'\right] - k_1[\Gamma_3(\Delta)Y' + \vartheta_1\Phi_1]\right\}$$

$$\Phi_6 = \frac{1}{hh_7}\left\{h\left[\left(B_{i+3}\Delta + \frac{v_i h_1}{h}\right)\Phi_{i+1} + \Gamma_3(\Delta)\nabla\cdot W'\right] - h_1[\Gamma_3(\Delta)Z' + \vartheta_2\Phi_1]\right\}$$

$$\Phi_7 = \frac{1}{k}[(\Delta + \tilde{\mu}_4^2)[\Gamma_3(\Delta)Y' + \vartheta_1\Phi_1 - \xi_i\Phi_{i+1}] + k_3\Phi_5]$$

$$\Phi_8 = \frac{1}{h}[(\Delta + \tilde{\mu}_5^2)[\Gamma_3(\Delta)Z' + \vartheta_2\Phi_1 - v_i\Phi_{i+1}] + h_3\Phi_6]$$

$$\Phi' = \frac{1}{\mu}\left[\Gamma_3(\Delta)H' - (\lambda' + \mu)\nabla\Phi_1 + \tilde{\lambda}_i\nabla\Phi_{i+1}\right]$$

$$\Phi'' = \frac{1}{k_6}\left[\Delta^2 \prod_{\substack{i=1\\4}} (\Delta + \tilde{\mu}_i^2)V' - \Delta(k_4 + k_5)\nabla\Phi_5 + B_i\Delta(\Delta + \tilde{\mu}_4^2)\nabla\Phi_{i+1} - k_1\nabla\Phi_7\right]$$

$$\Phi''' = \frac{1}{h_6}\left[\Delta^2 \prod_{\substack{i=1\\3,5}} (\Delta + \tilde{\mu}_i^2)W' - \Delta(h_4 + h_5)\nabla\Phi_6 + B_{i+3}\Delta(\Delta + \tilde{\mu}_5^2)\nabla\Phi_{i+1} - h_1\nabla\Phi_8\right]$$

$$\hat{\Phi}(x) = (\Phi', \Phi'', \Phi''', \Phi_2, \Phi_3, \Phi_4, \Phi_7, \Phi_8), \Lambda(\Delta) = \left(\Lambda_{gl}(\Delta)\right)_{14\times14},$$

$$\Lambda_{pp}(\Delta) = \Delta\Gamma_3(\Delta) = \Delta^2 \prod_{\substack{i=1\\3}} (\Delta + \tilde{\mu}_i^2), \qquad \Lambda_{p+3;p+3}(\Delta) = \Delta^2 \prod_{\substack{i=1\\4,6}} (\Delta + \tilde{\mu}_i^2),$$

$$\Lambda_{p+6;p+6}(\Delta) = \Delta^2 \prod_{\substack{i=1\\3,5,7}} (\Delta + \tilde{\mu}_i^2), \qquad \Lambda_{p+9;p+9}(\Delta) = \Gamma_3(\Delta) = \Delta \prod_{\substack{i=1\\3}} (\Delta + \tilde{\mu}_i^2),$$

$$\Lambda_{13;13}(\Delta) = \Delta^2 \prod_{\substack{i=1\\4}} (\Delta + \tilde{\mu}_i^2), \qquad \Lambda_{14;14}(\Delta) = \Delta^2 \prod_{\substack{i=1\\3,5}} (\Delta + \tilde{\mu}_i^2),$$

$$\Lambda_{gl}(\Delta) = 0, p = 1,2,3, g, l = 1,\dots,14, g \neq l$$

Appendix H

$$Z_{ij}(D_x) = \frac{1}{\mu}\Gamma_3(\Delta)\delta_{ij} + m_{11}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j}, Z_{i;j+3}(D_x) = m_{12}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

$$Z_{i;j+6}(D_x) = m_{13}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j}, Z_{i;p+8}(D_x) = m_{1;p+2}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{i+3;j}(D_x) = 0, Z_{i+3;j+3}(D_x) = \frac{1}{k_6}\Delta^2 \prod_{\substack{i=1\\4}} (\Delta + \tilde{\mu}_i^2)\delta_{ij} + m_{22}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

$$Z_{i+3;j+6}(D_x) = Z_{i+3;q}(D_x) = 0, Z_{i+3;13}(D_x) = m_{27}(\Delta)\frac{\partial}{\partial x_i}, Z_{i+6;j}(D_x) = 0,$$

$$Z_{i+6;j+3}(D_x) = 0, Z_{i+6;j+6}(D_x) = \frac{1}{h_6}\Delta^2 \overset{\substack{i=1\\3,5}}{\Pi} (\Delta + \tilde{\mu}_i^2)\delta_{ij} + m_{33}(\Delta)\frac{\partial^2}{\partial x_i\,\partial x_j},$$

$$Z_{i+6;e}(D_x) = 0, Z_{i+6;14}(D_x) = m_{38}(\Delta)\frac{\partial}{\partial x_i}, Z_{y+8;i}(D_x) = m_{y+2;1}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{y+8;i+3}(D_x) = m_{y+2;2}(\Delta)\frac{\partial}{\partial x_i}, Z_{y+8;i+6}(D_x) = m_{y+2;3}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{y+8;p+8}(D_x) = m_{y+2;p+2}(\Delta), Z_{13;i}(D_x) = Z_{13;i+6}(D_x) = Z_{13;q}(D_x) = 0,$$

$$Z_{13;i+3}(D_x) = m_{72}(\Delta)\frac{\partial}{\partial x_i}, \qquad Z_{13;13}(D_x) = m_{77}(\Delta),$$

$$Z_{14;i}(D_x) = Z_{14;i+3}(D_x) = Z_{14;e}(D_x) = 0, \qquad Z_{14;i+6}(D_x) = m_{83}(\Delta)\frac{\partial}{\partial x_i},$$

$$Z_{14;14}(D_x) = m_{88}(\Delta), \qquad m_{11}(\Delta) = -\frac{1}{\mu\tilde{C}}\left\{(\lambda' + \mu)M_{11}^*(\Delta) - \tilde{\lambda}_i M_{1;i+1}^*(\Delta)\right\},$$

$$m_{12}(\Delta) = \frac{\Delta + \tilde{\mu}_6^2}{kk_7\tilde{C}}[-k_1\vartheta_1 M_{11}^*(\Delta) + (B_i k\Delta + k_1\xi_i)M_{1;i+1}^*(\Delta)],$$

$$m_{13}(\Delta) = \frac{\Delta + \tilde{\mu}_7^2}{hh_7\tilde{C}}[-h_1\vartheta_2 M_{11}^*(\Delta) + (B_{i+3}h\Delta + h_1 v_i)M_{1;i+1}^*(\Delta)],$$

$$m_{1;y+2}(\Delta) = \frac{M_{1y}^*(\Delta)}{\tilde{C}}, \qquad m_{17}(\Delta) = \frac{1}{kk_7\tilde{C}}\left[(k_7\Delta - k_2)(\vartheta_1 M_{11}^*(\Delta) - \xi_i M_{1;i+1}^*(\Delta)) + k_3 B_i \Delta M_{1;i+1}^*(\Delta)\right],$$

$$m_{18}(\Delta) = \frac{1}{hh_7\tilde{C}}\left[(h_7\Delta - h_2)(\vartheta_2 M_{11}^*(\Delta) - v_i M_{1;i+1}^*(\Delta)) + h_3 B_{i+3}\Delta M_{1;i+1}^*(\Delta)\right],$$

$$m_{2z}(\Delta) = 0, \qquad m_{22}(\Delta) = -\frac{\Gamma_3(\Delta)}{kk_6k_7}[(k_4 + k_5)k\Delta + k_1k_3], \qquad m_{27}(\Delta) = \frac{k_3}{kk_7}\Gamma_3(\Delta),$$

$$m_{3s}(\Delta) = 0, \qquad m_{33}(\Delta) = -\frac{\Gamma_3(\Delta)}{hh_6h_7}[(h_4 + h_5)h\Delta + h_1h_3], \qquad m_{38}(\Delta) = \frac{h_3}{hh_7}\Gamma_3(\Delta),$$

$$m_{y+2;1}(\Delta) = -\frac{1}{\mu\tilde{C}}\left\{(\lambda' + \mu)M_{y1}^*(\Delta) - \tilde{\lambda}_i M_{y;i+1}^*(\Delta)\right\},$$

$$m_{y+2;2}(\Delta) = \frac{\Delta + \tilde{\mu}_6^2}{kk_7\tilde{C}}[-k_1\vartheta_1 M_{y1}^*(\Delta) + (B_i k\Delta + k_1\xi_i)M_{y;i+1}^*(\Delta)],$$

$$m_{y+2;3}(\Delta) = \frac{\Delta + \tilde{\mu}_7^2}{hh_7\tilde{C}}[-h_1\vartheta_2 M_{y1}^*(\Delta) + (B_{i+3}h\Delta + h_1 v_i)M_{y;i+1}^*(\Delta)], m_{y+2;r+2}(\Delta) = \frac{M_{yr}^*(\Delta)}{\tilde{C}},$$

$$m_{y+2;7}(\Delta) = \frac{1}{kk_7\tilde{C}}\left[(k_7\Delta - k_2)[\vartheta_1 M_{y1}^*(\Delta) - \xi_i M_{y;i+1}^*(\Delta)] + k_3 B_i\Delta M_{y;i+1}^*(\Delta)\right],$$

$$m_{y+2;8}(\Delta) = \frac{1}{hh_7\tilde{C}}\left[(h_7\Delta - h_2)\left[\vartheta_2 M_{y1}^*(\Delta) - v_i M_{y;i+1}^*(\Delta)\right] + h_3 B_{i+3}\Delta M_{y;i+1}^*(\Delta)\right],$$

$$m_{7z}(\Delta) = m_{8s}(\Delta) = 0, m_{72}(\Delta) = -\frac{k_1\Gamma_3(\Delta)(\Delta + \tilde{\mu}_6^2)}{kk_7}, m_{77}(\Delta) = \frac{\Gamma_3(\Delta)(k_7\Delta - k_2)}{kk_7},$$

$$m_{83}(\Delta) = -\frac{h_1\Gamma_3(\Delta)(\Delta + \tilde{\mu}_7^2)}{hh_7}, m_{88}(\Delta) = \frac{\Gamma_3(\Delta)(h_7\Delta - h_2)}{hh_7}$$

$i, j = 1,2,3, p = 2, \ldots, 6, q = 10,12,14, e = 10, \ldots, 13, y, r = 2,4, z = 1,3,6,8, s = 1,2,4,7$

Appendix I

$$\hat{\varsigma}_g(x) = -\frac{e^{\iota\tilde{\mu}_g|x|}}{4\pi|x|}, \qquad g = 1,\dots,7,$$

$$r'_{11} = -\sum_{p=1}^{3}\left(\prod_{j=1,j\neq p}^{3} \tilde{\mu}_j^2\right)\prod_{i=1}^{3} \tilde{\mu}_i^{-4}, \qquad r'_{12} = \prod_{i=1}^{3} \tilde{\mu}_i^{-2}, \qquad r'_{1;y+2} = \tilde{\mu}_y^{-4}\prod_{i=1,i\neq y}^{3} \left(\tilde{\mu}_i^2 - \tilde{\mu}_y^2\right)^{-1},$$

$$r'_{21} = -\sum_{p=1}^{4,6}\left(\prod_{j=1,j\neq p}^{4,6} \tilde{\mu}_j^2\right)\prod_{i=1}^{4,6} \tilde{\mu}_i^{-4}, \qquad r'_{22} = \prod_{i=1}^{4,6} \tilde{\mu}_i^{-2}, \qquad r'_{2;q+2} = \tilde{\mu}_q^{-4}\prod_{i=1,i\neq q}^{4,6} \left(\tilde{\mu}_i^2 - \tilde{\mu}_q^2\right)^{-1},$$

$$r'_{31} = -\sum_{p=1}^{3,5,7}\left(\prod_{j=1,j\neq p}^{3,5,7} \tilde{\mu}_j^2\right)\prod_{i=1}^{3,5,7} \tilde{\mu}_i^{-4}, \qquad r'_{32} = \prod_{i=1}^{3,5,7} \tilde{\mu}_i^{-2}, \qquad r'_{3;z+2} = \tilde{\mu}_z^{-4}\prod_{i=1,i\neq z}^{3,5,7} (\tilde{\mu}_i^2 - \tilde{\mu}_z^2)^{-1},$$

$$r'_{41} = \prod_{i=1}^{3} \tilde{\mu}_i^{-2}, \qquad r'_{4;y+1} = -\tilde{\mu}_y^{-2}\prod_{i=1,i\neq y}^{3} \left(\tilde{\mu}_i^2 - \tilde{\mu}_y^2\right)^{-1},$$

$$r'_{51} = -\sum_{p=1}^{4}\left(\prod_{j=1,j\neq p}^{4} \tilde{\mu}_j^2\right)\prod_{i=1}^{4} \tilde{\mu}_i^{-4}, \qquad r'_{52} = \prod_{i=1}^{4} \tilde{\mu}_i^{-2}, \qquad r'_{5;r+2} = \tilde{\mu}_r^{-4}\prod_{i=1,i\neq r}^{4} (\tilde{\mu}_i^2 - \tilde{\mu}_r^2)^{-1},$$

$$r'_{61} = -\sum_{p=1}^{3,5}\left(\prod_{j=1,j\neq p}^{3,5} \tilde{\mu}_j^2\right)\prod_{i=1}^{3,5} \tilde{\mu}_i^{-4}, \qquad r'_{62} = \prod_{i=1}^{3,5} \tilde{\mu}_i^{-2}, \qquad r'_{6;s+2} = \tilde{\mu}_s^{-4}\prod_{i=1,i\neq s}^{3,5} (\tilde{\mu}_i^2 - \tilde{\mu}_s^2)^{-1},$$

$y = 1,2,3,\ q = 1,4,6,\ z = 1,5,7,\ r = 1,\dots,4,\ s = 1,3,5$