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RESEARCH ARTICLE

Transverse free vibration of nanobeams with intermediate support using nonlocal strain gradient theory

Ufuk Gul

Trakya University, Department of Mechanical Engineering, Edirne, Türkiye

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Abstract

In the present study, the transverse free vibration of intermediately supported nanobeams is investigated in the framework of size-dependent nonlocal strain gradient theory. Unlike the classical elasticity theory, nonlocal strain gradient theory considers the micro/nano scale effects in mechanical analysis of size-dependent structures. The potential and kinetic energies have been derived for the nanobeam model and the Ritz method has been used to determine the natural frequencies of intermediately supported nanobeams. A cantilever nanobeam model with intermediate support is considered in the analysis. By changing the position of the intermediate support, dimensionless vibration frequencies of the nanobeam model have been obtained. Obtained results from the present nonlocal strain gradient theory showed that hardening or softening material responses have been observed according to the classical elasticity theory depending on the magnitude of the material length scale parameter and nonlocal scale parameter. This mechanical behavior can provide an advantage to designers while modeling the micro/nanostructures. Present results can be used for the design of nano-sensors, nanotube-based resonators, and oscillators.

1. Introduction

With superior mechanical, electrical, and optical properties of nanotube structures have been used in many areas, like the aerospace industry, nanomedicine, energy storage, and semiconductor materials. Nano-sized structures such as nanobeams have great attention due to their wide applications in nanoelectromechanical systems (NEMS) [1]. Examples of applications of nanobeams are oscillators and resonators [2], nano-actuators [1], atomic force microscopy [3], and sensors [4,5]. Because of this, investigating the mechanical behaviors of these nanostructures is of great significance for better understanding or designing nano-scaled systems and devices. To investigate the static and dynamic behaviors of these nano-scale structures, molecular dynamic simulations [6], experiments [7,8], and some higher-order size-dependent continuum theories are generally used. Molecular dynamic simulations and experiments give satisfactory results in static and dynamic analyses of nanostructures. However, molecular dynamic simulations are time-consuming due to the complicated simulation process and experiments have some difficulties in controlling and applying at the nano-scale. Accordingly, size-dependent higher-order continuum theories, such as nonlocal elasticity theory [9], strain gradient theory [10], couple stress theory [11], and doublet mechanics [12] can be applied to static and dynamic analyses of nanostructures.

Correspondence Ufuk Gul ufukgul@trakya.edu.tr

Nonlocal elasticity theory takes into account the inter-atomic long-range interactions and Eringen [9] showed that axial wave dispersion curves in nonlocal elasticity theory are matched with the atomic lattice dynamics model. Another mostly used size-dependent continuum theory is known as strain gradient elasticity theory which is introduced by Mindlin [10]. In this theory, stress is a function of strain and its gradient. The nonlocal elasticity theory and strain gradient elasticity theory predict different material responses. Nonlocal elasticity theory predicts softening material behavior whereas strain gradient models predict hardening material behavior with respect to classical elasticity theory. It is known from the experiments [13,14] that material behaviors become softening or hardening depending on the microstructure of the considered material. In this context, Lim et al. [15] introduced the nonlocal strain gradient elasticity theory which includes both strain gradient and nonlocal parameters in the constitutive equations. Nonlocal strain gradient theory predicts both softening and hardening material responses like experimental studies. Based on the nonlocal strain gradient elasticity theory, the static and dynamic analyses of nanobeams are investigated in the open literature.

Ebrahimi and Barati [16] investigated the vibration of axially functionally graded nanobeams under thermal and magnetic field effects using nonlocal strain gradient theory. Both softening and hardening effects on the vibration of axially functionally graded nanobeams have been shown in that study. Li and Hu [17] studied wave propagation in fluid-conveying viscoelastic carbon nanotubes via nonlocal strain gradient elasticity theory. In another study [18], transverse wave propagation of embedded S-FGM nanobeams has been examined based on nonlocal strain gradient theory. The same authors [19] examined the wave propagation in quasi-3D functionally graded nanobeams in a thermal environment considering the nonlocal strain gradient theory. They analytically obtained the wave frequency, phase velocity, and group velocity of functionally graded nanobeams. Li et al. [20] investigated the wave propagation analysis of viscoelastic carbon nanotubes with surface effect under magnetic field using nonlocal strain gradient theory. Wave dispersion relations have been obtained and the effect of magnetic field on wave propagation of carbon nanotubes has been discussed in that paper. Based on the nonlocal strain gradient theory, axial vibration of axially functionally graded nanobeams resting on an elastic foundation has been studied in [21]. Hamilton's principle is used to derive the governing equations of the nonlocal strain gradient nanobeam model and the Galerkin solution has been employed to obtain the natural frequencies of nanobeams. Lu et al. [22] developed a unified nonlocal strain gradient model for bending and buckling of nanobeams using nonlocal strain gradient theory. They showed that softening or hardening material behaviors occur depending on the magnitude of the material length scale parameter and nonlocal parameter in the nonlocal strain gradient model. Transverse free vibration of axially moving nanobeams has been investigated with nonlocal strain gradient theory in [23]. The complex modal analysis method is employed to solve the governing equation of the considered nanobeam model. Rajasekaran and Khaniki [24] developed a finite element formulation for static and dynamic analyses of non-uniform nanobeams based on the nonlocal strain gradient theory. Thermal buckling and free vibration of functionally graded nanobeams have been investigated within the framework of the nonlocal strain gradient theory in [25]. Merzouki et al. [26] proposed a new finite element formulation for nonlocal strain gradient nanobeam models using trigonometric two-variable shear deformation theory. Static and free vibration analyses have been executed in that paper. Furthermore, there are some important studies related to static and dynamic analyses of nanobeam structures based on nonlocal strain gradient theory [27-33].

In this article, free vibration of cantilever nanobeam with intermediate support has been investigated within the framework of the nonlocal strain gradient theory, in which the stress accounts for not only the nonlocal elastic stress field but also the strain gradient stress field. Ritz solution technique is performed to obtain an approximate solution to the free vibration analysis of nanobeam with intermediate support. Natural frequencies obtained from the present model are compared to the classical elasticity solution. The influences of the position of the intermediate support, material length scale parameter, and nonlocal parameter on the free vibration of nanobeam are discussed in detail. The qualitative results obtained from the present study can provide a reference for the mechanical design and processing of carbon nanotube-based structures.

2. Nonlocal strain gradient theory

The integral form of the nonlocal stress tensor σ_{ij} and nonlocal higher-order stress tensor $\sigma_{ijm}^{(1)}$ can be determined as [15]:

$$\sigma_{ij} = C_{ijkl} \int_V \alpha_0 \left(|x - x'|, e_0 a\right) \varepsilon'_{kl} dV \tag{1}$$

$$\sigma_{ijm}^{(1)} = \tau^2 C_{ijkl} \int_V \alpha_1 \left(|x - x'|, e_1 a\right) \varepsilon'_{kl,m} dV \tag{2}$$

where ε'_{kl} and $\varepsilon'_{kl,m}$ are the strain and higher-order strain tensors, respectively,e_0 and e_1 are nonlocal scale parameters, a is the internal characteristic length which is the bonding length between two atoms in the considered material, τ is the material length scale parameter of the strain gradient model, C_{ijkl} is the fourth-order elasticity tensor, $\alpha_0(|x - x'|, e_0 a)$ and $\alpha_1(|x - x'|, e_1 a)$ are the nonlocal kernel functions for the classical and higher-order stress tensors, respectively. The total stress tensors of the nonlocal strain gradient theory account for both nonlocal stress and strain gradient stress tensors as below

$$t_{ij} = \sigma_{ij} - \nabla \sigma_{ijm}^{(1)} \tag{3}$$

where t_{ij} is the total stress and ∇ is the Laplacian operator. For simplicity, assuming $e_0 = e_1 = e$ and discarding terms of $O(\nabla^2)$ the general constitutive equation of nonlocal strain gradient elasticity theory is written as [33]:

$$(1 - (ea)^2 \nabla^2)\sigma_{ij} = C_{ijkl}(1 - \tau^2 \nabla^2)\varepsilon_{kl} \tag{4}$$

It should be noted that the general constitutive equation (4) incorporates the nonlocal parameter (ea) and material length scale parameter (τ). Ignoring the only nonlocal parameter $(ea = 0)$ in Eq. (4) leads to the strain gradient theory and ignoring the only material length scale parameter $(\eta = 0)$ in Eq. (4) leads to the nonlocal elasticity theory. If one neglects both nonlocal parameter (ea) and material length scale parameter (τ) in Eq. (4) constitutive equation of classical elasticity theory can be obtained.

3. Ritz method

The approximate Ritz method is used to analyze the transverse free vibration of nanobeam with intermediate support based on the nonlocal strain gradient theory. A cantilever nanobeam with intermediate support with length L is seen in Fig. 1. The position of the intermediate support is described as lL.

Accordingly, the potential energy of the nanobeam can be written as [15]:

$$U = \frac{1}{2}\int_0^L EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}\int_0^L EI\, \tau^2 \left(\frac{\partial^3 w}{\partial x^3}\right)^2 dx \tag{5}$$

where U denotes the potential energy, w denotes the transverse displacement, E denotes the elasticity modulus, I denotes the moment of inertia and τ denotes the material length scale parameter is related to the strain gradient model. The kinetic energy for the nanobeam model can be expressed as [15]:

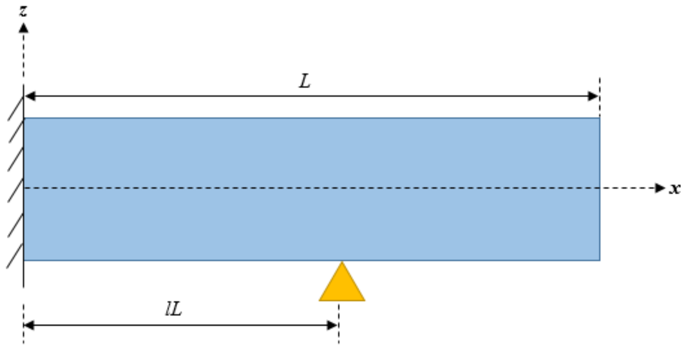

Fig. 1. A nanobeam model with intermediate support

$$T = \frac{1}{2}\int_0^L \rho A \left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L \rho A\,(ea)^2 \left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 dx \tag{6}$$

where T denotes the kinetic energy, A denotes the cross-sectional area, ρ denotes the density, ea denotes the nonlocal scale parameter and t denotes the time. Assuming harmonic motion, the transverse displacement $w(x,t)$ of the nanobeam model can be defined as

$$w = \overline{W}(\bar{x}) sin\omega t \tag{7}$$

where ω is the natural frequency and $\overline{W}(\bar{x})$ is the dimensionless displacement function. Inserting Eq. (7) into Eqs. (5) and (6) yield the maximum potential energy and maximum kinetic energy of the nanobeam model in the following form

$$U_{max} = \int_0^1 \left(\frac{\partial^2 \overline{W}}{\partial \bar{x}^2}\right)^2 d\bar{x} + \int_0^1 \eta^2 \left(\frac{\partial^3 \overline{W}}{\partial \bar{x}^3}\right)^2 d\bar{x} \tag{8}$$

$$T_{max} = \int_0^1 \lambda^4 (\overline{W})^2\, d\bar{x} + \int_0^1 \lambda^4 \mu^2 \left(\frac{\partial \overline{W}}{\partial \bar{x}}\right)^2 d\bar{x} \tag{9}$$

where λ is the dimensionless frequency of the nanobeam, μ^2 is the dimensionless nonlocal parameter and η denotes the dimensionless material length scale parameter of the nanobeam. The dimensionless parameters in Eqs. (8) and (9) are described as follows:

$$\bar{x} = \frac{x}{L}, \overline{W} = \frac{w}{L}, \lambda^4 = \frac{\rho A \omega^2 L^4}{EI}, \mu^2 = \left(\frac{ea}{L}\right)^2, \eta^2 = \left(\frac{\tau}{L}\right)^2. \tag{10}$$

Then, the dimensionless displacement function can be defined in the following form for the Ritz method [34]:

$$\overline{W}(\bar{x}) = \sum_{i=i_0}^{I} D_i\, \psi_i(\bar{x}). \tag{11}$$

where D_i represents the unknown coefficients and $\psi_i(\bar{x})$ represents a function that satisfies at least the geometric boundary conditions of the nanobeam. The algebraic polynomial $\psi_i(\bar{x})$ can be assumed in dimensionless form as below

$$\psi_i(\bar{x}) = (\bar{x} - 0)^{a_1} (\bar{x} - l)^{a_2} (\bar{x} - 1)^{a_3}\, (\bar{x}^{i-1}) \tag{12}$$

For the present study, natural frequencies of the nanobeam model will be obtained for the clamped-simply supported-free case. In this context, a_1, a_2 and a_3 parameters are taken as 2, 1, and 0, respectively. Thus, Eq. (12) turns into:

$$\psi_i(\bar{x}) = (\bar{x})^2(\bar{x} - l)(\bar{x}^{i-1}) \tag{13}$$

To obtain the natural frequencies of the nanobeam with intermediate support, the following function is defined:

$$F = U_{max} - T_{max} \tag{14}$$

Then, Eq. (14) should be minimized with respect to unknown coefficients (D_i) as follows:

$$\frac{\partial F}{\partial D_i} = 0 \tag{15}$$

Eq. (15) gives $I \times I$ simultaneous, linear, and homogeneous equations in an equal number of D_i unknowns. These obtained equations can be defined as an eigenvalue problem for dimensionless vibration frequencies of nanobeam with intermediate support:

$$([K] - \lambda^2[M])\{\Delta\} = 0 \tag{16}$$

Here, λ^2 is the eigenvalues (natural frequencies) of the nanobeam with intermediate support, $[K]$ and $[M]$ are the stiffness and mass matrices, respectively, and $\{\Delta\}$ is the column vector of the unknown coefficients. Also, mode shapes can be found by substituting the obtained natural frequencies into Eq. (15) and solving for the eigenvector constituents D_i/D_1.

4. Numerical Results

In this section, numerical examples of transverse free vibration of nanobeam with intermediate support are presented. Nonlocal strain gradient theory is used in the formulation and the dimensionless frequencies (λ) of nanobeams are obtained by the approximate Ritz method.

In Table 1, a convergence study on the Ritz method is presented for the first four modes of vibration on classical ($\mu = \eta = 0$) clamped-free (C-F) and clamped-simply supported (C-S) beams. It is seen that the solution of the Ritz method agrees well with the exact solution [35] and a negligible difference occurs between the two solution methods when I = 25. Thus, I = 25 terms can be taken in the Ritz method for vibration solution of the nanobeam model in the remaining part of the present article.

Comparison of the first three dimensionless frequency parameters (λ^2) is presented based on the nonlocal strain gradient model in Table 2. The frequencies obtained from the present study are in good agreement with Ref. [27]. The highest difference between the two studies is obtained for the third mode of vibration (1.55%). It is noted that this acceptable difference is due to the solution method.

Table 1. Validation of Ritz solution with analytical results (I = 25)

	C-F ($\mu = \eta = 0$)			C-S ($\mu = \eta = 0$)		
Mode number	Ritz solution	[35]	Difference%	Ritz solution	[35]	Difference%
1	1.8854	1.8751	0.55	3.9320	3.9266	0.14
2	4.7201	4.6940	0.55	7.0783	7.0685	0.14
3	7.8985	7.8547	0.55	10.2243	10.2101	0.14
4	11.0571	10.9955	0.56	13.3704	13.3517	0.14

Table 2. Comparison of C-F nanobeam model for the first three dimensionless frequency parameters (λ^2) (L = 5 nm)

Mode number	C-F ($\tau^2 = (ea)^2 = 0.1$ nm^2) Present study	[27]	Difference%
1	3.5291	3.5125	0.47
2	22.0167	21.8220	0.89
3	61.4277	60.4891	1.55

Table 3. The effect of nonlocal strain gradient theory on dimensionless vibration frequencies (I = 25)

		$l = 0.1$			$l = 0.5$			$l = 1$		
μ	η	m = 1	m = 2	m = 3	m = 1	m = 2	m = 3	m = 1	m = 2	m = 3
0	0	2.0300	5.0930	8.5370	3.1428	7.8634	9.4399	3.9320	7.0783	10.2243
	0.1	2.6968	7.4284	14.3770	4.0805	12.2782	19.5460	5.6752	11.7868	19.5713
	0.2	2.9432	8.3589	16.6463	4.4750	14.0385	22.9951	6.3887	13.6605	22.9752
0.1	0	1.8162	3.4829	4.9260	2.4926	4.5186	5.6472	3.2407	4.6717	5.7060
	0.1	2.3122	4.7887	8.1154	3.1031	6.6656	10.7838	4.5009	7.6033	10.7977
	0.2	2.4983	5.3813	9.4371	3.3750	7.6411	12.6668	5.0472	8.8227	12.6879
0.2	0	1.6793	3.0614	4.2315	2.2120	3.8881	4.8446	2.9079	4.0323	4.8642
	0.1	2.0999	4.2257	7.0047	2.7287	5.7788	9.1858	3.9960	6.5702	9.2165
	0.2	2.2610	4.7599	8.1520	2.9634	6.6334	10.7895	4.4770	7.6287	10.8316

The nonlocal strain gradient effect on vibration frequencies for different positions of the intermediate support is illustrated in Table 3. It is seen that softening or hardening material behaviors occur with respect to classical elasticity theory ($\mu = \eta = 0$) depending on the magnitude of the material length scale parameter (η) and nonlocal parameter (μ). The hardening effect on the nanobeam is more pronounced by increasing the dimensionless material length scale parameter (η) which is related to strain gradient theory. On the other hand, softening material response becomes more apparent with increasing the dimensionless nonlocal parameter (μ). Dimensionless frequencies increase when intermediate support approaches the free end at the first three modes in nonlocal and nonlocal strain gradient theories. However, only the first mode of vibration frequencies increases with approaching the intermediate support to the free end and the other second and third modes of vibration frequencies decrease with approaching the intermediate support to the free end of the nanobeam in the strain gradient theory. The reason for this behavior may be related to nodal points of neighboring mode numbers.

Fig. 2 shows the variation in frequency ratio ($\lambda/\lambda_{classical}$) with mode numbers for different positions of intermediate support. The ratio of nonlocal frequencies, strain gradient frequencies, and nonlocal strain gradient frequencies to classical elasticity frequencies can be seen for the first five modes in this figure. Accordingly, nonlocal elasticity theory predicts softening material response whereas strain gradient theory predicts hardening material response with respect to the classical elasticity theory. The nonlocal strain gradient theory predicts softening or hardening material behavior depending on the magnitude of the material length scale parameter and nonlocal parameter. Generally, when the material length scale parameter is higher than the nonlocal parameter ($\mu = 0.1, \eta = 0.2$) hardening material response occurs. However, softening material response occurs where the nonlocal parameter is higher than the material length scale parameter in the nonlocal strain gradient theory.

The variation of the dimensionless frequencies of nanobeam with the position of the intermediate support is shown in Fig. 3. It is seen that vibration frequencies increase while intermediate support approaches the free end of the beam at the first mode. This behavior is also seen in the second and third modes for the nonlocal and nonlocal strain gradient elasticity theories. However, second and third mode frequencies firstly increase, then decrease a little bit when intermediate support approaches the free end for the classical and strain gradient theories. Moreover, the highest and lowest frequency results are obtained for the strain gradient theory and nonlocal theory, respectively.

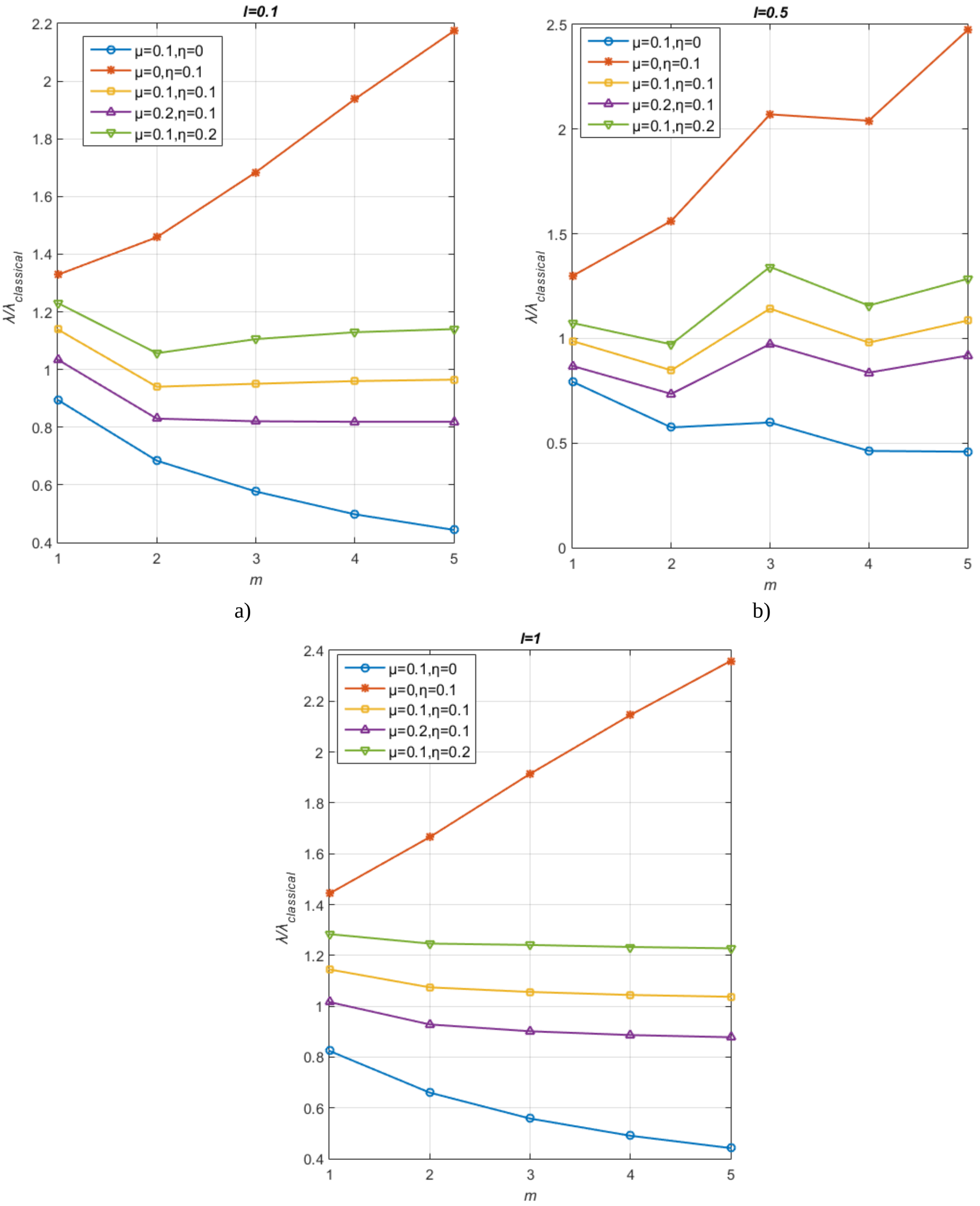

Fig. 2. Variation of dimensionless frequency ratio with the mode number (*m*)

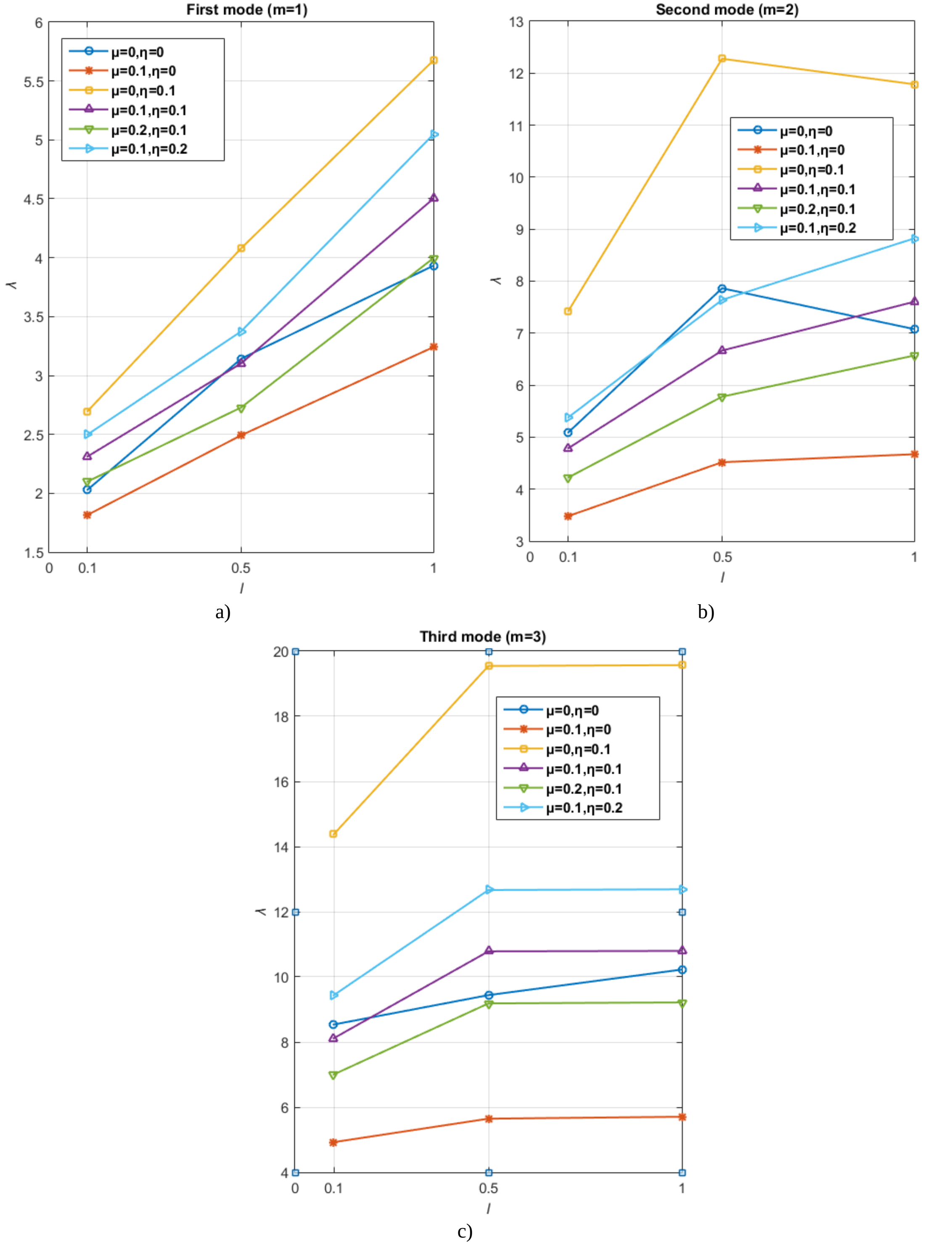

Fig. 3. Variation of the dimensionless frequency with the position of the intermediate support

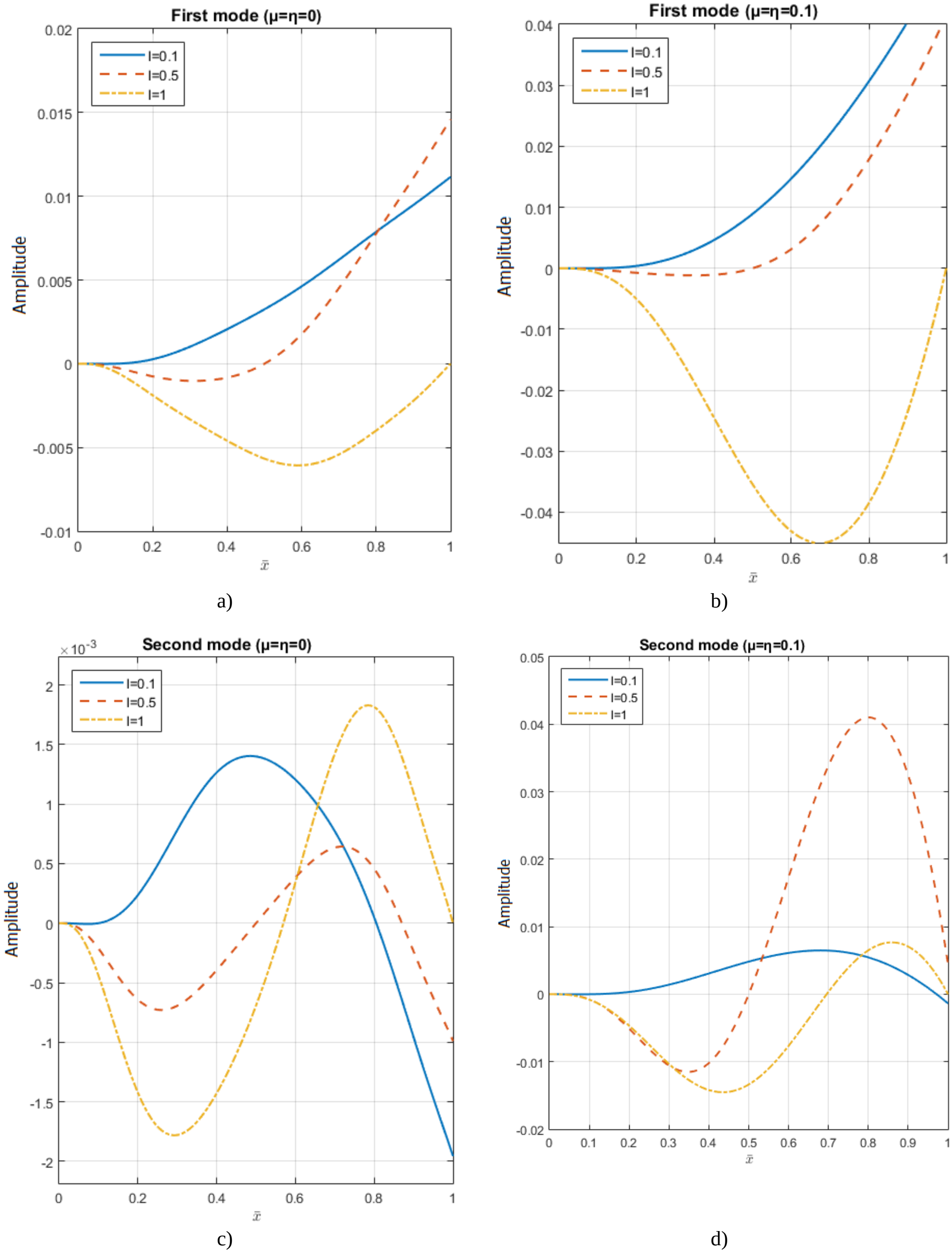

Fig. 4. Mode shapes of nanobeam with intermediate support

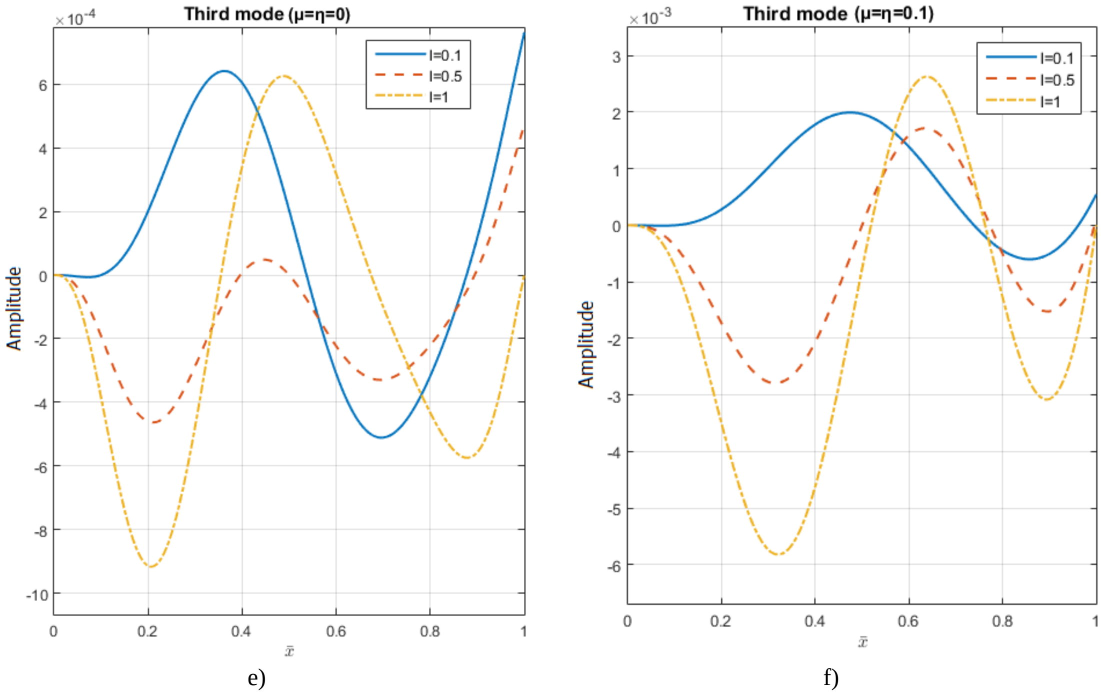

Fig. 4. Cont'd

In Fig. 4, mode shapes of nanobeam with intermediate support at the first three vibration modes are shown for classical and nonlocal strain gradient theories. Nonlocal strain gradient theory increases the amplitude of the displacements in mode shapes. Depending on the nodal points of neighboring mode numbers, the position of the intermediate support increases or decreases the dimensionless frequencies of nanobeam.

5. Conclusions

This paper investigates the free vibration of cantilever nanobeam with intermediate support based on the size-dependent nonlocal strain gradient theory. After obtaining the minimum total potential and kinetic energies of the nanobeam, the Ritz method has been applied to find natural frequencies. Softening or hardening material responses have been obtained in nonlocal strain gradient theory. These material behaviors (softening or hardening) are changing with respect to the magnitude of the nonlocal and material length scale parameters in nonlocal strain gradient theory. The material behaviors seen in this theory can provide significant advantages in the mechanical designs of nanostructures. The position of the intermediate support in the nanobeam affects the vibration frequencies. Vibration frequencies increase when intermediate support approaches the free end of the beam for nonlocal and nonlocal strain gradient theories. However, this behavior is seen only at the first mode frequencies, and at the second and third mode frequencies firstly increase, then decrease a little bit when intermediate support approaches the free end for strain gradient theory. Also, the mode shapes of the nanobeam are directly affected by the position of the intermediate support and the amplitude of the displacements in mode shapes increases with the nonlocal strain gradient model.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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